Observing Changes in the Max Box with a Graphic Calculator

Application: Changing function values

Objective: **Develop a class that causes students to independently solve the minimums and maximums of textbook problems. Use the calculator to provide interesting information and stimulate thinking.**

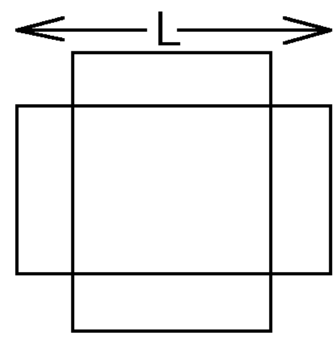
The Max Box is a common problem found in most textbooks.

Though everyone tends to think that the graphic calculator can be used to produce a quick answer, this is not necessarily the case. The teaching of mathematics is certain to change in the future as students use the graphic calculator and everyone evaluates individual answers obtained to discover the whether or not they are correct. This exercise can be termed a "discovery teaching method."

The following problem is typical of those found in many textbooks.

Problem 1

There is a cardboard square with the length of the sides measuring *L* cm. If we cut out four identical squares (length of side = *x*cm) from the four corners of the square, we can fold the remaining leaves up to form an open box. How long should the length of the sides of the four cutout squares be in order to maximize the volume of the box?



When posing this problem to students, the teacher might suggest substituting the student's seat number for the value of *L* to determine the volume. In a class of forty students, this would create 40 separate problems. This gives each student the responsibility to solve their own individual problem. Then the teacher can lead the students through a conversation that stresses the individual traits of students, such as "Student A has strong mathematical sense, Student B is good at calculation, and Student C hates mathematics."

Teacher: If the value of *L* is your seat number and the length of the sides of the cutout squares is *x*, what formula can we write for volume *y*?

Student A: From the diagram, volume *y* is the area of the base times the height. So, isn't the formula:

$$y = x\,(L - 2x)^2,\ 0 < x < \frac{L}{2} \qquad (L\text{: Student seat number}) \ldots\ldots\ldots\ldots\ldots(1)$$

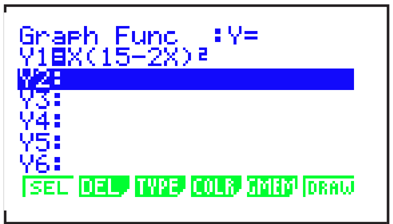
Teacher: Then what?

Student A: Then, from this formula, can we determine the value of *x* that produces the greatest volume (*y*). In short, a table of increases and decreases.

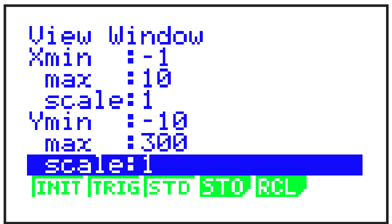
Teacher: So, we'll make a table of increases and decreases that we learned about before, listing the maximum area and length of the sides of the cutout squares for each student in the table on the blackboard.

Thus, the students begin to analyze their tables of increases and decreases on the graphic calculators, employing a variety of different methods.

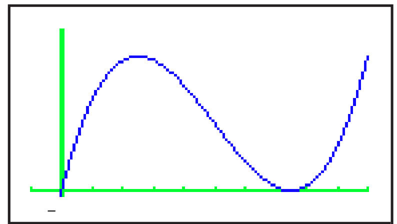
Student A (Seat No. 15)



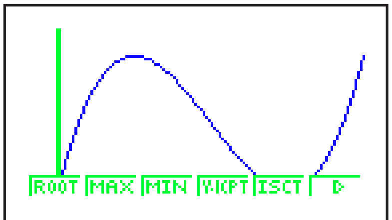
Select the Graph menu from the MAIN MENU, enter $x(15–2x)^2$ for Y1, and select F6 (DRAW).



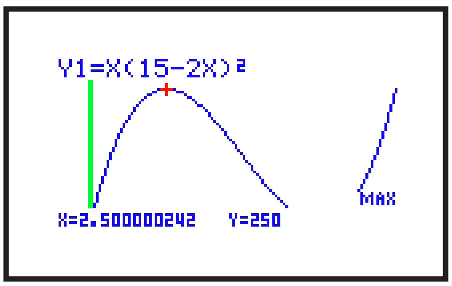
Setup the screen with SHIFT F3 (V-WIN). Press EXE after settings are completed.



Press F6 (DRAW) to draw the graph. Next, press SHIFT F5 (G-SLV).



Press F2 (MAX) to determine the maximum value.



Student A said, “I got it.” He rounded the length of the sides of the cutout square to one decimal place, and wrote x = 2.5 for the length of one side of the squares and maximum volume of y = 250 on the blackboard.

A salute to the power of the graphic calculator.

Student C gave Student A a smug look for immediately developing the procedure that produced the answer.

Teacher: Wait a minute. You’re not done yet. Fully differentiate equation (1) and write a table of increases and decreases.

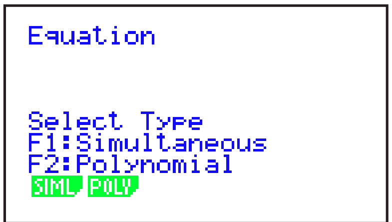
Output of Student B (Seat No. 28)

First, differentiation of $y = (28–2x)^2$ is required. (The method for differentiating a product is not generally known at this level of high school.)

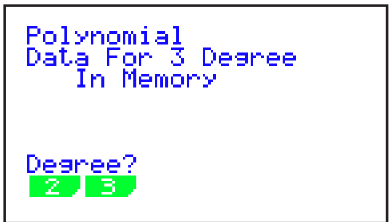
Differentiation of $y = 4x^3–112x^2+784x$ yields $y’=12x^2–224x+784$.

Now the graphic calculator can be used to investigate the maximum value at $y’ = 0$.

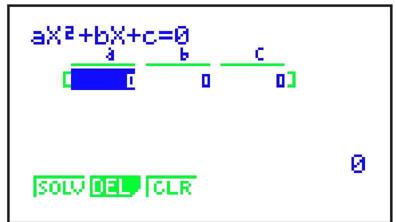
Method for solving a quadratic equation



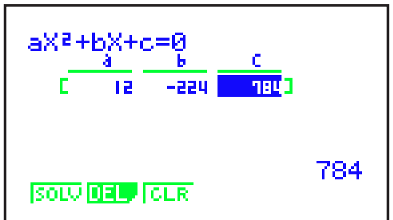
Select the Equation Mode from the MAIN MENU, and then press [F2] (POLY).



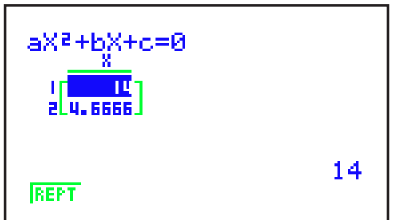
Press [F1] (2) for quadratic equation.



The equation is solved by entering the coefficients and pressing [F1] (SOLV).



12 → [EXE] → –224 → [EXE] → 784 → [EXE] → [F1] (SOLV).



The solutions are 14 and 4.66.

Completed table of increases and decreases

x	...	4.67	...	28
y’	+	0	–	/
y	↗	Maximum	↘	/

Note: Cutting out squares with sides of 14 (cm) from the square (L=28) will not produce a box. The length of the cutout square is 4.67 (cm), and the maximum volume is 1,625 (cm^3). (Rounded to two decimal places)

Teacher: Student B’s method is fine, but… What about everyone else?

Students: Student A, Student A

Teacher: Let’s fill in the table on the blackboard.

L (Seat No.)	1	2	3	4	5	6	7	8	9	10	11	...
x	0.17	0.3	0.5	0.67	0.83	1	1.17	1.33	1.5	1.67	1.83	...
y max	0.1	0.6	2	12	9.3	16	25.4	75	54	74.1	98.6	...

Teacher: I want you to investigate the characteristics (of the equation) from these tables, the relationship between the length of the square (L) and maximum volume (y max).

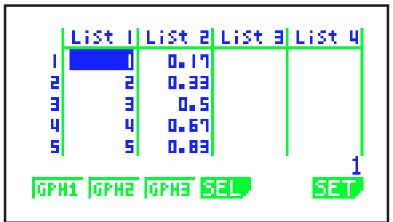
Student C: Can we plot the table as a graph with the graphic calculator?

Teacher: Of course. First, should we examine the relationship between the length of the sides of the square (L) and the length of the sides of the cutout squares (x)?

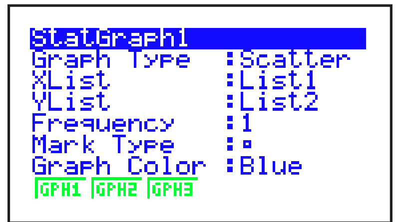
Am I the only one who thinks that graphic calculators can be used skillfully in certain situations?

Method for Plotting a List

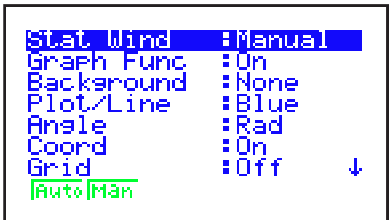
Select the Statistics Mode from the MENU, and input Seat No. L into List 1 and cutout square side length (x) into List 2. Then, press [F1] (GRPH).



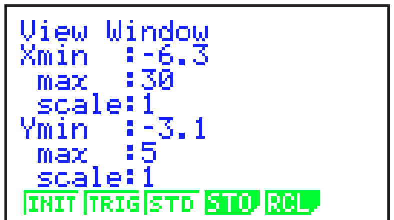
Examine the maximum values of List 1 and List 2, and then press [F6] (SET).



Press [F1] (Scat) for the Graph Type. Press [F1] (List1) for Xlist1 and [F2] (List2) for the Ylist.



Press [SHIFT] [MENU] [F2] (Man) and change Stat Wind from Auto to Manual.

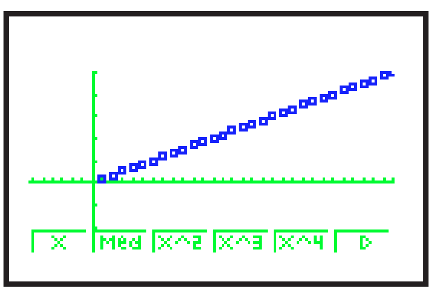


Press [SHIFT] [F3] (V-WIN) and then input the maximum value of List1 for Xmax and the maximum value of List2 for Ymax with. Press [EXE] ⟶ [F1] (GRAH) ⟶ [F1] (GPH1) to produce the graph.

In Case of Problems

(1) Mistaken input of list items can be immediately corrected through re-input of the items.

(2) To delete a list item, press MENU and select LIST. Then, delete the item with DEL ([F3]). Press MENU, and select STAT to return input of list items.



Teacher: How about it? This is the plot of your table.

Student A: It appears to be a straight line.

Student C: It's a straight line that's not perfectly straight.

Teacher: That's true. But, the input data is rounded to two decimal places, so the line should not be perfectly straight.

Student A: It must be a straight line. I'd like to know the equation.

Teacher: It's not a perfectly straight line, so investigate using values that you know are correct. Take two natural numbers from the table and determine the equation that produces a straight line through those two points. Don't use the graphic calculator!

Student B: A straight line through two points (6,1) and (12,2) is produced by $y-1=\frac{2-1}{12-6}(x-6)$, which can be simplified to $y=\frac{1}{6}x$.

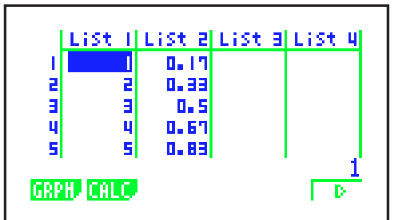
Teacher: That appears to be the case. I think the correct formula is $x=\frac{1}{6}L$.

Student A: The value of the cutout square is one sixth of side L?

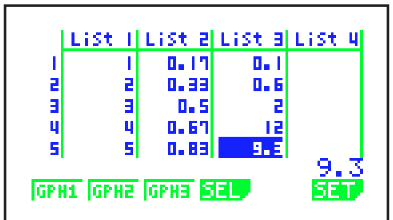
Student C (Seat No. 8) immediately divided 8 by six to produce a value of 1.333.

Student B: We should also understand the relationship between the length of side L and maximum volume (*y max*).

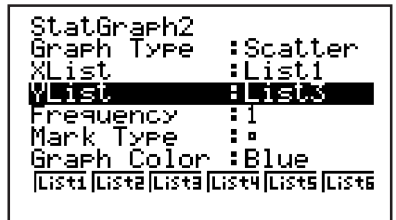
Teacher: Let's check the list again.



Select the Statistics Mode from the MENU, input the maximum volume to (*y max*) toLIST3, and press F1 (GRPH).



Examine the maximum values of List 1 and List 3, and then press F6 (SET).



Select F2 (GPH2) to change to StatGraph2.
Select F3 (List3) for YList and press EXE → SHIFT → F3 (V-WIN).

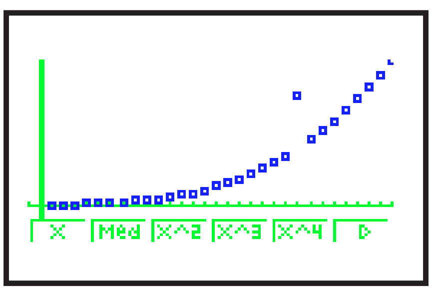
Input the maximum value of List1 into Xmax and the maximum value of List3 to Ymax. Then press EXE → F1 (GRAH) → F2 (GPH2) to draw the graph.

In Case of Problems

(1) If the initial screen (list table) does not appear, press SHIFT → AC/ON to turn off power. Press AC/ON again to turn power back on, and select STAT from the MAIN MENU.

(2) If the previous graph appears on the screen, change the Ylist setting in line 3 of the StatGraph2 screen (step 3 above) from List 2 to List 2, by pressing F3. Next, press EXE and then F2 (GPH2) to draw the graph shown below.

(3) If you cannot see the part of the graph you want, use the cursor keys ( ) to scroll the screen in the desired direction.



Student B: There it is! It's a curve.

Student C: It's a curve, but one point falls outside the curve.

Student A: It must be a mistake. It's not reasonable that only this point is an exception.

Teacher: The horizontal axis indicates that it's student (seat) no. 22. Re-check that value.

No. 22: That's right. It's not 1,500; 789 is the correct value.

Teacher: Are there any other mistakes? Let's set the ranges of the *x* and *y* axes on the screen (third screen), and take a look. (There are two mistakes near the origin - no. 4 and no. 8)

Student A: Is this curve quadratic? Cubi? What degree is it?

Teacher: You can't always produce a formula from a graph. This is where you need math ability. A formula that determines volume *y* from the side of the square (value of the seat number) is just the beginning.

Student B: From the discussion of volume, we can use student A's formula (1).

$$y = x\,(L-2x)^2,\ 0 < x < \frac{L}{2} \qquad \text{(L: Seat No.)} \;\ldots\ldots\ldots\ldots\ldots\ldots \; (1)$$

Student A: In that case, if L is a number, we then have a simple cubic function. Therefore, we can differentiate the function and determine the extreme values.

$$y' = 12x^2 - 8Lx + L^2 = (6x-L)(2x-L) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \; (2)$$

The conditions for x yield an extreme value of $x = \frac{L}{6}$.

Student C: There are two extreme values for equation (2). Why consider only one?

Student A: The other extreme value doesn't produce a box. We can also say this based on formula (1). We can also confirm the formula is $x = \frac{L}{6}$ from the formula for the straight line on our original graph.

Student B: Then, from the derived curve, we can substitute $x = \frac{L}{6}$ in formula (1).

Thus, $\frac{L}{6}\left(L - 2\frac{L}{6}\right)^2 = \frac{2}{27}L^3$ is the graph of a curve.

Teacher: We have reached a conclusion, so let's take notes.

> **Today's Discovery:** The length of the sides of four identical squares cut out from the four corners of a square of length L must be $\frac{L}{6}$ to maximize the volume of the resulting box. The maximum possible volume is $\frac{2}{27}L^3$.

This is an example of a discovery learning process that employed graphic calculators. Repetitive calculations were done on the calculator, and student interest was maintained by a learning method that allowed understanding of the characteristics of the problem from tables and graphs. More importantly, their "thinking ability" is enhanced by graphic calculators.