A Tinsmith Takes Care of Business with a Graphic Calculator

Application: Changing of function values

Objective: **The previous problem presented material that stimulated rapid development of student abilities. While the classroom presentation of this problem is similar to Max Box, the problem is approached from the opposite perspective. Have some fun using the standpoint of a tinsmith.**

Min Tin is an optimization problem that is a continuation of the Max Box problem. The “Tin” of Min Tin is the tin of a tin can, because the problem deals with minimizing the surface area of a tin can without changing the volume. This problem was prepared using parts of a problem introduced in Mathematics in School (Volume 22, 1993) by G. Corris of England’s Millfield School. The conversation between the teacher and students is shown to enhance understanding.

Teacher: A tinsmith received this facsimile.

Fax (Problem) Send 100 cm^2 copperplate measuring boxes (each with a volume of 500cm^3) as soon as possible (any dimensions acceptable).

The tinsmith very happy when he saw this fax. He cut four identical squares from the corners of a copperplate to form a 500-cm^3 measuring box. What were the dimensions of the 100 measuring boxes that the tinsmith made?

Student A: I think it’s easier to think about one measuring box rather than 100 boxes all at once. We may be able to understand this problem if we think of it in terms of how long to make the side of the square (x) in order to make a measuring box with a volume of 500 cm^3.

Student C: To make a measuring box from a square of side x, we have to cut out the four corners. We can treat this problem like the previous Max Box problem.

Teacher: What is the formula if the length of the cutout square is H?

Student B: The height of measuring box (H) created from a square of length x yields a formula for its volume of $V = (x - 2H)^2 H$.

Student A: According to the fax, the volume is given. V = 500 cm^3 is a constant, and Student B’s formula becomes:

$$500 = (x - 2H)^2 H,\ x > 2H \qquad \text{.....................} (1)$$

Student C: That equation can’t be solved. We have two variables and one equation. (Brief period of silence)

Student A: If we assign an arbitrary value to H, then can write a quadratic equation and solve for x.

Teacher: OK. Let H be your seat number, and each of you solve your own quadratic equation for x. It might be interesting. It will give us some practice in quadratic equations.

Student B: Expanding formula (1) and writing a quadratic equation for x is easy. The formula becomes:

$$x^2 - 4Hx + 4H^2 - \frac{500}{H} = 0 \qquad \text{.....................} (2)$$

This is the solution of equation (2).

Teacher: Like the Max Box problem, the student solving the problem fills in the table on the blackboard.

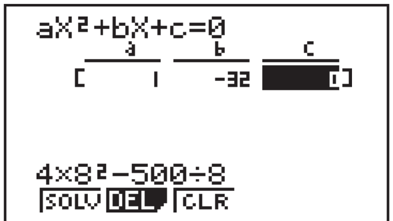
Student C: I'm Number 8, so I substitute 8 for H in equation (2) and solve the quadratic equation for x.

Thus, the solution to $x^2 - 4 \bullet 8x + 4 \bullet 8^2 - \dfrac{500}{8} = 0$ is...

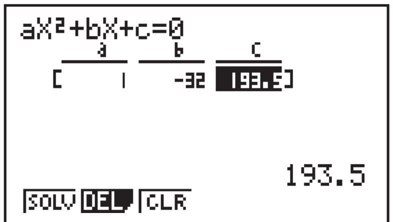
Teacher: This is exactly the time to use the functions of the graphic calculator to produce the solution.
See “The method for solving a quadratic equation,” which was introduced in the Max Box problem, for the operation sequence.

The complex calculations required for solving the problem are left to the calculator. This learning method is one way to relieve the drudgery of mathematics for students like Student C, who “hates” math.

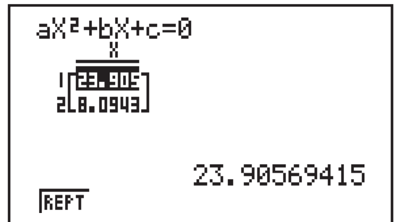
The operation sequence for solution of the quadratic equation in the Max Box problem calls for selection of the setting screen for a quadratic equation.



Focus on the equation in the lower left of the screen. You can input the calculation formula for each coefficient.



Press F1 (SOLV) after all coefficients are input as shown here.



The two solutions to the quadratic equation are 23.9 and 8.09.

Student C: When the length of each side of the cutout square is 8 cm, the length of each side of the original square must be 8.09 or 23.9 (cm) to produce a measuring box that has a volume of 500 cm^3.

Student B: The 8.09 value solves the mathematical equation, but you cannot have a cutout square that is 8 cm in length if the original square has a length of 8.9 cm.

Student A: Look at formula (1). Failure to satisfy the condition $x > 2H$ means that a measuring box cannot be formed. Therefore, the only solution to student C’s equation is 23.9 cm.

Teacher: Make this kind of table.

Table 1 - Different 500cm^3 measuring boxes of each student

H: Seat No.	1	2	3	4	5	6	7	8	9	10	11	12	13	...
x	24.4	19.8	18.9	19.2	20	21.1	22.5	23.9	25.5	27.1	38.7	30.5	32.2	...

Student C: It’s surprising that so many measuring boxes with different dimensions can have the same volume.

Teacher: Interesting! Which of the measuring boxes did the tinsmith make?

Student B: The tinsmith was happy because he could make the 500cm^3 square measuring boxes of any dimensions.

Student C: We should determine which measuring box produces the lowest material costs.

Student A: Oh?! Certainly the tinsmith picked the measuring box with the smallest surface area.

Student B: Surface area *A* can be written as $A = x^2 - 4H^2$, so we should substitute the derived *x* and *H* values in this formula.

Teacher: OK. Add the “surface area” row to the above table.

Table 2 - Output of Surface Areas (produced immediately by the calculator)

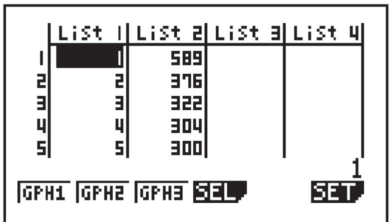
H: Seat No.	1	2	3	4	5	6	7	8	9	10	11	12	13	···
x	24.4	19.8	18.9	19.2	20	21.1	22.5	23.9	25.5	27.1	38.7	30.5	32.2	···
Surface area A	589	376	322	304	300	302	308	315	324	333	342	352	361	···

Student C: We should choose the measuring box in the table that has the smallest surface area.

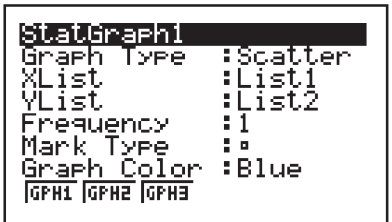
Student A: Let's plot the list as a graph on the graphic calculator.

Initialize the screen as shown below, as learned in the previous Max Box exercise, and input the list.

Method for Plotting a List



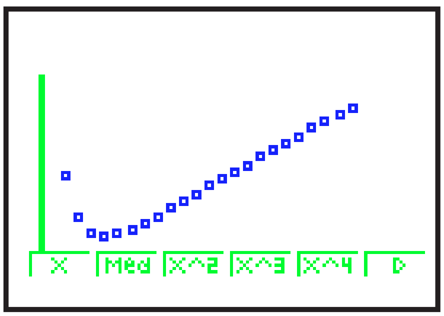
Input the seat numbers to List 1 and the surface areas to List 2. Then, press F1 (GRPH) → F6 (SET).



Press F1 (Scat) for the Graph Type. Press F1 (List 1) for the XList and F2 (List 2) for the YList, then press EXE → F2 → F3 (V-WIN).

Set the minimums and maximums for both axes as shown on the screen. Then, press EXE → F1 (GRPH) → F1 (GPH1) to produce the graph.

(Figure 1)



In Case of Problems

(1) If the minimum value is not on the graph, use the cursor keys ( ) to scroll the graph in to the position you want.

(2) When an attempt to input List1 or List2 is unsuccessful because data I already input in these lists, delete the data for List1 by pressing MENU → F4 (LIST) → F4 (DEL-A) → F1 (YES). Delete the data for List2 using the same procedure. Input can then be accomplished on this screen. Press MENU → 2 (STAT) after input to return to the initial screen.

Student C: From the graph, it looks like the surface area is minimized by the fourth value (No. 4).

Student A: The table shows the fifth value (No. 5).

Teacher: The surface area for the first value is 589, so it is not displayed on the above graph. Look - the maximum value for the *y*-axis on the graph setting screen is 500.

Student B: Then, the tinsmith made a measuring box with a surface area of 300cm^2 by cutting squares with 5cm long sides from a square copperplate?

Teacher: We can say that from the data. (See the reference section for details.)

Student C: What's the price of copperplate?

Teacher: Use 5 yen per cm^2.

Student B: The tinsmith's material costs are 1,500 yen (500 x 3) per measuring box. Therefore, his material costs are 150,000 yen (1,500 x 100) for 100 boxes.

Student A: The surface area of the box created from my seat number (15) is 380, so the total material costs are 380 x 5 x 100 = 190,000 yen. These boxes cost 40,000 yen more than the tinsmith's boxes.
Student B's (no. 28) measuring boxes cost 95,500 yen more than the tinsmith's boxes!

Student C: The tinsmith took care of business with a graphic calculator.

Reference

The Min Tin problem is a prime example of the applicability of mathematics to everyday life. Differential calculus can also be used to easily determine the minimum value of the surface area.

V = Surface area (A), H = Length of the cutout squares, x = Length of the original square

$V = (x-2H)^2 H$ .... (1) $A = x^2 - 4H^2$ ....(2)

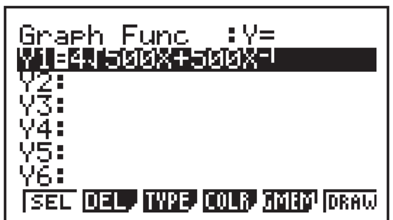
Solve formula (1) for x, $x = 2H + \sqrt{\dfrac{V}{H}}$ and substitute the result in formula (2) to eliminate x.

Thus, the formula becomes: $A = 4\sqrt{VH} + \dfrac{V}{H}$ .... (3)

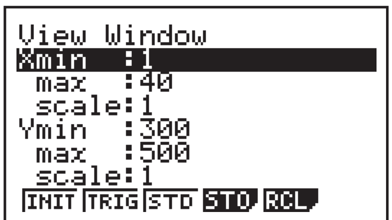
Formula (1) yields $\dfrac{dA}{dH} = 2\sqrt{\dfrac{V}{H}} - \dfrac{V}{H^2}$. Therefore, $\dfrac{dA}{dH} = 0$ when $H = \sqrt[3]{\dfrac{V}{4}}$.

Operating Procedure

(3) Express formula (3) as a graph when V = 500.

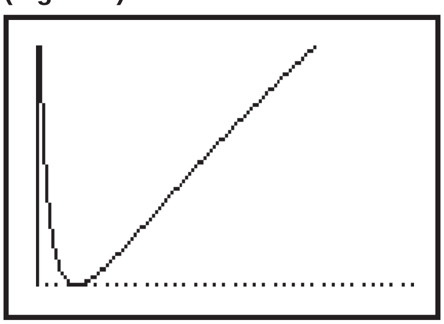


From the menu, enter GRAPH and input the formula. Next, press SHIFT → F3 (V-WIN) and make the settings shown.



Press EXE → F6 (DRAW) to draw the graph.

(Figure 2)



Generally, the length of each side of the cutout square can be determined from $H = \sqrt[3]{\dfrac{V}{4}}$. The length can be substituted in $x = 2H + \sqrt{\dfrac{V}{H}}$ to determine the minimum value (length) of the original square.

The graph (Figure 1) derived from Table 2 and the graph shown in Figure 2 are quite similar. This is an actual example of use of the graphic calculator to easily solve optimization problems without using higher mathematics.