Solving Problems with a Graphic Calculator

As shown in the diagram, G. Polya divided the problem solving process into four stages in his 1954 book titled “How to Solve It.” The problems in this section are different from the previous problems, which had set methods for solutions (algorithms). These problems force discovery of solution methods based on the problem. Complete understanding of the problem (Stage 1) is a prerequisite for solving these types of problems. A graphic calculator is used in Stage 1 to achieve speed and precision that is impossible with paper, pencils, and erasers. Graphic representation of problems aids understanding and provides hints for deriving solutions. Use of the graphic calculator to derive formulas in Stage 2 also allows greater recognition of the importance of mathematical expression of phenomena. Mathematical processing of problems in Stage 3 demands greater ability to think with mathematics, while Stage 4 presents the possibility of high-order understanding of problems or inducing general solutions toproblem. Thus, use of the graphic calculator in Stage 1 creates an environment for understanding mathematics than could not be obtained from paper, pencils, and erasers. Problems for high school students have been selected that are appropriate for graphic calculators. The problems indicate that graphic calculators can be used for solution of the problems, and sample solutions are also shown.

Stage 1 - Understanding the problem
Stage 2 - Planning the solution
Stage 3 - Executing the plan
Stage 4 - Testing the solution

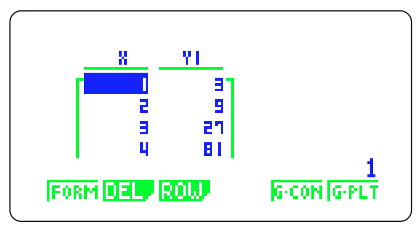
(1) Integer problems

Problem 1: **Determine the units digit of 3^{333}.**

<Understanding the problem using a graphic calculator>

Select TABLE from the Menu and press 3 [^] [X,θ,T] [EXE] to input $Y1 = 3^\wedge x$. Next, press [F5] (RANG) to set the range (Start, End) and pitch of x as shown in Figure 1. Press [EXE] twice to obtain the number table for 3^n (n = 1, 2, ··· 10) shown in Figure 2.

(Figure 1)



(Figure 2)

<Understanding the problem and planning the solution>

It can be determined from Figure 2 that the units digits of 3^n (n = 1, 2, ···) are 3, 9, 7, 1, 3, 9, 7, 1, ... and 3, 9, 7, and 1, repeating in that sequence. Therefore, it can be inferred that a units digit of 3^{333} can be determined by the remainder produced by dividing the exponent 333 by 4. The solution can be obtained if this relationship is expressed mathematically.

Solution

If the last digit of 3^n is expressed as $C_{(n)}$, then $3^{n+4} = 3^n \bullet 3^4 = 3^n(80+1) = 3^n \bullet 80 + 3^n$ produces: $C^{(n+4)} = C_{(n)}$

Therefore, $C(333) = C(82 \times 4 + 1) = C(1) = 3$ .... (solution)

Comments

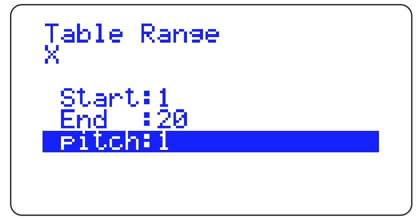
When a is a positive integer, the units digit of a^n (n=1, 2, ···) becomes $A_{(n)}$.
Derivation of a $A_{(n)}$ from a=1, 2, 3, ··· is also fully feasible.

Problem 2: **Determine the positive integers that produce a remainder of 2 when divided by 3, a remainder of 1 when divided 5, and a remainder of 2 when divided by 7.**

<Understanding the problem using a graphic calculator>

Select TABLE from the MENU and press 3 [X,θ,T] [+] 2 to input $Y1 = 3x + 2$. Input $Y2 = 5x + 1$ in the same manner, and press [F5] (RANG) to set the range of x as shown in Figure 3. Press [EXE] twice to display the values for X, $Y1$, and $Y2$ as shown in Figure 4. The table shows that the common values of $Y1$ and $Y2$ are 11, 26, 41, 56 ... This means the formula for the common values of $Y1$ and $Y2$ can be expressed as:

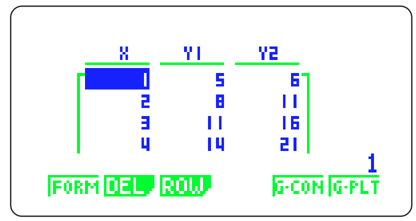
$$y = 11 + 15(x - 1)$$



(Figure 3)

Now input $Y3 = 15x - 4$ and $Y4 = 7x + 2$, and press [F1] (SEL) to produce the display shown in Figure 5 showing $Y3$ and $Y4$ only. Press [EXE] to display the tables of $Y3$ and $Y4$ shown in Figure 6. The only common value of $Y3$ and $Y4$ is 86 due to the Table Range (Figure 3). Change the End of the Table Range to 30 to produce two common values, 86 and 191. This means, the integer problem can be represented as:

$$86 + 105(x - 1)$$

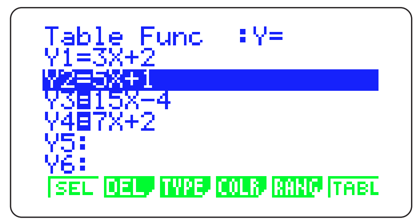


(Figure 4)

Solution

Express the integers that can produce a remainder of 1 when divided by 5 as $5m + 1$. All m are integers, so substitute $3k$, $3k + 1$, and $3k + 2$ for m in $5m + 1$ to yield:

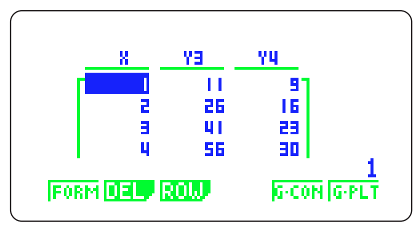
$$15k + 1,\ 5(3k + 1) + 1,\ 5(3k + 2) + 1\ (k \text{ is an integer})$$



(Figure 5)

Within this series, the values that produce a remainder of 2 when divided by 3 are $15k + 11$. In the same manner, when $k = 7\ell$, $7\ell + 1$, $7\ell + 2$, $7\ell + 3$, $7\ell + 4$, $7\ell + 5$, $7\ell + 6$ are substituted in $15k + 11$, the values that produce a remainder of 2 when divided by 7 are $k = 7\ell + 5$. Therefore, series of integers can be expressed as:

$$15(7\ell + 5) + 11 = 105\ell + 86\ (\ell = 0,1,2 \cdots) \quad \text{....... (solution)}$$



(Figure 6)

Comments

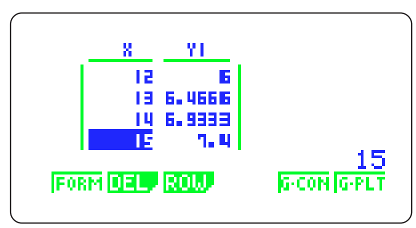
1. The Table Range must be set to a much wider range and the calculation becomes more complex if $Y1 = 3x + 2$, $Y2 = 5x + 1$, $Y3 = 7x + 2$ are input simultaneously as the calculator operation.
2. The second calculator operation is the method to determine the common values of $Y3 = 15x - 4$ and $Y4 = 7x + 2$. Solving $Y3 = 15x - 4$ for $x \pm y$ yields, which can be expressed as:

$$15y - 4 = 7x + 2,\ y = \frac{7x + 6}{15}$$

The integer values of y when $x = 1, 2, ...$ can be extracted from the X and Y tables shown in Figure 7.

6, 13, 20,...

This series is produced from $k = 7\ell + 6$, so it can be inferred that the integers can be determined from $15(7\ell + 6) - 4$.



(Figure 7)

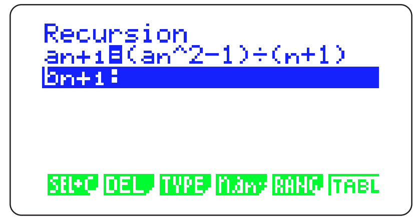
Note: The following problems include “understanding the problem and planning the solution” within “understanding the problem using a graphic calculator.”

(2) Progression problems

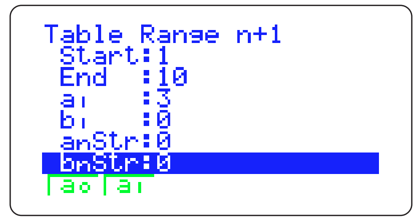
Problem 3: **Express the progression defined by $a_1 = 3,\ a_n{}^2 = (n+1)a_n + 1\ (n \geqq 1)$ as the general case of $\{a_n\}$ in terms of n.**

<Understanding the problem using a graphic calculator>

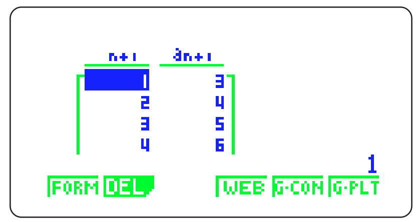
Press [F3] (TYPE) to select RECUR (progression) from the MENU and press [F2]. Next, input [(] [F4] (n, a_n) [F2] (a_n) [^] [2] [−] [1] [÷] [(] [F1] (n) [+] [1] [)] [EXE] to display the screen shown in Figure 8. Press [F5] (RANG) and input the table range from n = 1 to n = 10 with an initial value of a_1 = 3 as shown in Figure 9. Then, press [EXE] twice to produce the tables for n + 1 and a_n + 1. Thus, it can be surmised that the relationship between n and a_n can be expressed as $a_n = n + 2$.



(Figure 8)



(Figure 9)



(Figure 10)

Solution

$a_n = n + 2$ can be inferred from $a_1 = 3$, $a_2 = 4$, $a_3 = 5$.

i) The progression holds for $a_1 = 1 + 2 = 3$ when $n = 1$.

ii) Assuming that the progression holds when $n = k$, the formula becomes

$a_k = k + 2$ .... (1)

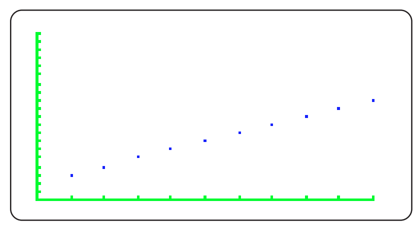
Then, from $a_{n+1} = \dfrac{a_n{}^2 - 1}{n+1}$ and (1)

$$\therefore a_{k+1} = \frac{a_k{}^2 - 1}{k+1} = \frac{(k+2)^2 - 1}{k+1} = \frac{(k+1)(k+3)}{k+1} = k + 3 = (k+1) + 2$$

Thus, the progression also holds when $n = k + 1$.
This means from i) and ii) that the progression for all natural numbers of n is $a_n = n + 2$ .... (solution).

Comments

Pressing [F6] can be pressed from Figure 10 to produce a graph of the relationship between n and a_n as shown in Figure 11. This graph makes interpretation easier. At that time, press [SHIFT] [F3] (V-WIN), and set Xmin (minimum value of x) = 0, Xmax (maximum value of x) = 10, scale = 1, Ymin (minimum value of Y) = 0, Ymax (maximum value of Y) = 20, and scale = 1.



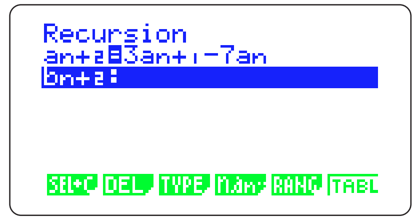
(Figure 11)

Problem 4: **A progression of integers $\{a_n\}$ is defined by the recursion formula $a_1 = 1, a_2 = 3, a_{n+2} = 3a_{n+1} - 7a_n (n = 1, 2, \ldots)$.**
Determine all values of n that produce even numbers for $\{a_n\}$.

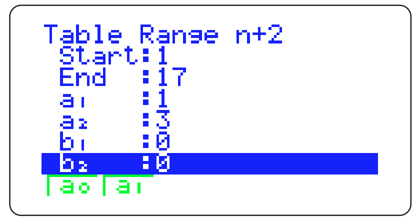
<Understanding the problem using a graphic calculator>

Press [F3] (TYPE) to select RECUR (progression) from the MENU, then select [F3] ($a_{n+2}=Aa_{n+1}+Ba_n+\cdots$). Press [F4] ($n$, a_n) [3] [F3] (a_{n+1}) [−] [7] [F2] (an) [EXE] to display the formula as shown in Figure 12. Press [F5] (RANG) to set the table range as shown in Figure 13. Then press [EXE] twice to display the tables for $n + 2$ and a_{n+2} as shown in Figure 14. (Press the cursor down key ( ) twice in succession.)

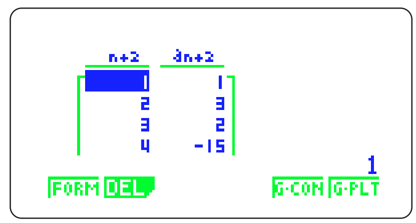
From Figure 14, it can be seen that an produces the even numbers 2, -72, 1906, -43,920, 927,362,··· when n = 3, 6,9, 12, 15,··· All other results are odd numbers. These facts and the given recursion formula can be used to derive the solution.



(Figure 12)



(Figure 13)



(Figure 14)

Solution

$a_3 = 2,\ a_6 = -72,\ a_9 = 1906$ .... (1)

Then, $a_{n+2} = 3a_{n+1} - 7a_n$ .... (2)

Therefore,

$$a_{n+3} = 3a_{n+2} - 7a_{n+1}$$

$$a_{n+3} = 3(3a_{n+1} - 7a_n) - 7a_{n+1} \quad (\because\ (2))$$

$$= 2a_{n+1} - 21a_n$$

$$= 2(a_{n+1} - 11a_n) + a_n \quad \text{........... (3)}$$

Because a_{n+1} and a_n are integers, $2(a_{n+1} - 11a_n)$ produces even numbers.
If a_n are defined as even numbers, then from (3) a_{n+3} also produces even numbers. (4)
Therefore, from (1) and (4),
a_n is an even number when $n = 3m$ (m = 1, 2,···) .... (solution)

Comments

Changing the form of the expression to (3) became the solution method because it can be assumed that a_{3m} is even and a_{3m+1} and a_{3m+2} are odd. Then, from $a_{3m+3} = 3a_{3m+2} - 7a_{3m+1}$ both $3a_{3m+2}, -7a_{3m+1}$ must produce odd numbers. Thus, we can also say that a_{3m+1} represents non-odd numbers, which is to say even numbers.