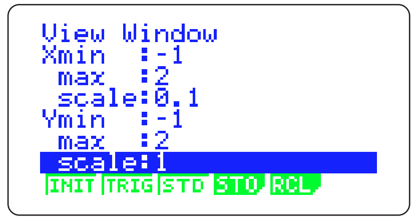
(3) Differentiation and integration

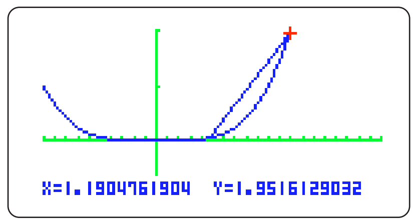
Problem 5: **Use the properties of the graph of $\int(x) = x^4 (x > 0)$ to prove the following inequality.**
Prove $\dfrac{a^4+b^4}{2'} \geqq \left(\dfrac{a+b}{2}\right)^4$ when $a > 0$, and $b > 0$.

<Understanding the problem using a graphic calculator>

Use MENU to select GRAPH, and input $Y1 = x \wedge 4$. Press SHIFT F3 (V-WIN) and then make the settings shown in Figure 15. Next, draw the graph. Press SHIFT F4, (SKTCH), F6 (▷) F2 (LINE) and use the cursor keys ( ) to move the pointer (+) to any point A of $Y1 = x \wedge 4 (x > 0)$. Press EXE and then use the cursor to move the pointer (+) to another point B of $Y1 = x \wedge 4 (x > 0)$. As shown in Figure 16, the graph of $f(x)$ lies below line segment AB except at endpoints A and B.
This means that the y-coordinate of the midpoint of line segment AB is clearly greater than the value of $f(x)$ obtained from the x-coordinate of the midpoint of line segment AB. The solution can be derived by expressing this relationship mathematically.



(Figure 15)



(Figure 16)

Solution

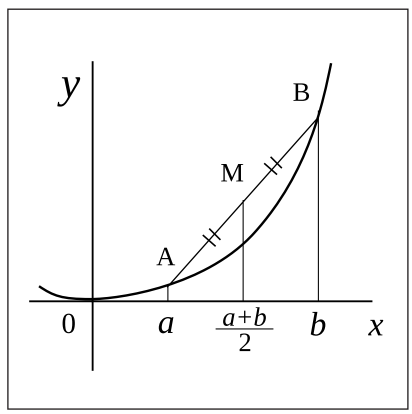
Derivatives $f'(x) = 4x^3$, $f''(x) = 12x^2$ obtained from $f(x) = x^4$ produce $f'(x) > 0$ and $f''(x) > 0$ when $x > 0$. Therefore, $f(x)$ is an convex increasing function when $x > 0$.

Thus, if $A(a, a^4)$, $B(b, b^4)$, $(0 < a < b)$, then the midpoint (M) of AB is:

$$M\left(\frac{a+b}{2}, \frac{a^4+b^4}{2}\right), \text{ and } \frac{a^4+b^4}{2} > \int\left(\frac{a+b}{2}\right)$$

Therefore, $\therefore \dfrac{a^4+b^4}{2} \geqq \left(\dfrac{a+b}{2}\right)^4$

Equality occurs when $a = b$. (The given formula also holds for the symmetric expression of a and b when $0 < b < a$.)



Comments

1. Even with $f(x) = x^n (n = 1, 2, \cdots) (x > 0)$, the proof for $\dfrac{a^n+b^n}{2} \geqq \left(\dfrac{a+b}{2}\right)^n$ also holds.
   This proof is much easier than mathematical induction.
2. If $f(x)$ is a convex function, then the following $pf(a) + (1-p)\ f(b) \geqq f(pa + (1-p)b)$ can be generalized for $0 < p < 1$.

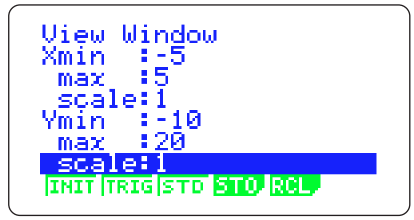
Problem 6: **For $f(x) = x^3 - 3x^2 - x + 3$, let minimum value be A, the maximum value be B, and the midpoint of AB be M to solve the following problems.**

(1) Show that point M lies on $f(x)$.

(2) When a straight line ℓ through point M intersects $f(x)$ at two points (C and D) in addition to point M, prove that the areas (S_1 and S_2) bounded by $f(x)$ and line segments CM and DM are equal.

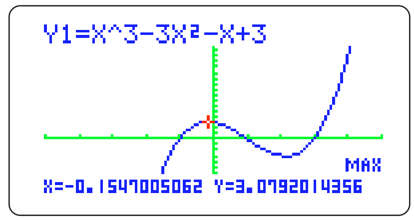
<Understanding the problem using a graphic calculator>

(1) Select GRAPH from the MENU, and input $Y1 = x\wedge3 - 3x^2 - x + 3$. Set the View Window as shown in Figure 17, and press [SHIFT] [F5] (G-SLV) [F2] (MAX) to have the pointer (+) indicate the maximum point on the graph screen and display X = -0.1547... and Y = 3.0792... as shown in Figure 18.



(Figure 17)

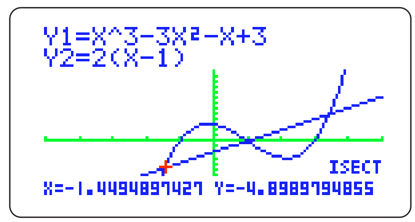
Return to the graph display screen, and press [SHIFT] [F5] (G-SLV) [F3] (MIN) to move the pointer (+) to the minimum point on the graph and display X = 2.1547... and Y = -3.0792... Midpoint M can be inferred to be point (1, 0) from the two graphs. The value of Y when X =1 can be confirmed by mental calculation of $Y1$ = 0 without selecting TABLE from the [MENU].



(Figure 18)

(2) Input one straight line ℓ as $Y2 = 2(x-1)$, and display the graphs for $Y1$ and $Y2$ on the same screen to show that S_1 and S_2 are equal. As a precaution, press [SHIFT] [F5] (G-SLV) [F5] (ISCT) to display the coordinates of intersection C (left endpoint) as shown in Figure 19.

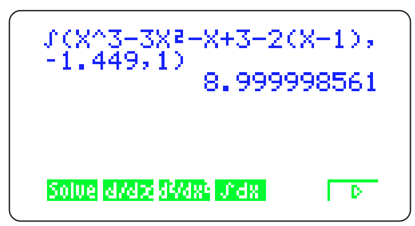
$$X = -1.4494\cdots,\ Y = -4.8989\cdots$$



(Figure 19)

Press the cursor right key (▶) to move the pointer (+) to point M, and display X = 1, Y = 0. Press the cursor right key (▶) again to read the coordinates (X = 3.4494..., Y = 4.8989...) of intersection D (right endpoint).

Next, select RUN from the MENU, and press [OPTN] [F4] (CALC) to select Integral dx ($\int dx$). Input $x\wedge3 - 3x^2 - x + 3 - 2(x-1)$ (the derived integration function), -1.449 (the lower limit), and 1 (the upper limit). Then press [EXE] to determine the integral value as shown in Figure 20. Input $x\wedge3 - 3x^2 - x + 3 - 2(x-1), 1, 3.449)$ as before, and press [EXE] to obtain -8.999. Although the sign of the previous value is different, the absolute values are equal, so you can confirm that $S_1 = S_2$.



(Figure 20)

All this means you must show that $S_1 - S_2 = 0$.

Solution

(1)

x	…	$1-\frac{2}{\sqrt{3}}$	…	$1-\frac{2}{\sqrt{3}}$	…
$f'(x)$	+	0	–	0	+
$f(x)$	↗	Maximum	↘	Minimum	↗

Then, we can rewrite the original function as:

$$f(x) = (3x^2 - 6x - 1)\left(\frac{1}{3}x - \frac{1}{3}\right) - \frac{8}{3}x + \frac{8}{3}$$

Therefore, $f\left(1 \pm \frac{2}{\sqrt{3}}\right) = \mp\frac{16\sqrt{3}}{9}$

Thus, maximum point A is $\left(1 - \frac{2}{\sqrt{3}}, \frac{16\sqrt{3}}{9}\right)$ and minimum point B is $\left(1 + \frac{2}{\sqrt{3}}, -\frac{16\sqrt{3}}{9}\right)$.

Consequently, midpoint M of AB is $M(1, 0)$, and from $f(1) = 0$ it follows that M lies on $f(x)$.

(2) Straight line ℓ, which passes through M and intersects $f(x)$ at two points (C and D) in addition to point M, is represented by:

$y = a(x - 1)$ .... (1) (a is a constant)

If we set the x-coordinates of intersections C and D of $f(x)$ and (1) to a and b ($a < b$), then the calculation of areas becomes:

$$S_1 - S_2 = \int_\alpha^1 \left\{x^3 - 3x^2 - x + 3 - a(x-1)\right\}dx - \int_1^\beta \left\{a(x-1) - (x^3 - 3x^2 - x + 3)\right\}dx$$

$$= \int_a^\beta \left\{x^3 - 3x^2 - x + 3 - a(x-1)\right\}dx$$

Then, $x^3 - 3x^2 - x + 3 = a(x-1), (x-1)(x^2 - 2x - 3 - a) = 0$

Therefore, from the solution of $x^2 - 2x - 3 - a = 0$, a and b are:

$\alpha = 1 - \sqrt{a+4}, \beta = 1 + \sqrt{a+4}$

Thus,

$$= \int_{1-\sqrt{a+4}}^{1+\sqrt{a+4}} \left\{(x-1)^3(a+4)(x-1)\right\} dx \quad (*)$$

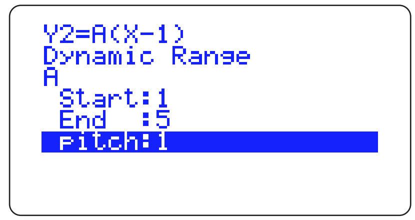
$$= \left[\frac{1}{4}(x-1)^4 - \frac{1}{2}(a+4)(x-1)^3\right]_{1-\sqrt{a+4}}^{1+\sqrt{a+4}}$$

$= 0$

$\therefore S_1 = S_2$

Comments

1. The calculation becomes very difficult if one does not realize that the formula can be rewritten in step (2). In that case, the proper calculator operation is to select (DYNA) from the MENU and input $Y2 = A(x-1)$ as $Y2$. Press F4 (VAR) F2 (RANG) to set the Dynamic Range as shown in Figure 21, then press EXE twice. "One Moment Please" will be displayed on the screen, and the slope of $Y2$ will change to 1, 2, 3, 4, 5, 4, 3, 2, 1, 2 ··· when the bar ▭ changes from white to black. From repeated operations it is possible that one will realize that point M is the point of symmetry for $f(x)$. If it can be inferred that M is the point of symmetry for $f(x)$, then at that time the point of symmetry (1,0) for $f(x)$ can be represented as $P'(X, Y)$.



(Figure 21)

Then, from $\dfrac{x+X}{2} = 1$, $\dfrac{y+Y}{2} = 0$, $x = 2 - X$, and $y = -Y$ can be substituted in $y = f(x)$. If $Y = f(X)$ can be shown, then $S_1 = S_2$ has been proven.

2. The polynomial function $f(x) = ax^3 + bx^2 + cx + d \, (a \neq 0)$ is symmetrical at points $\left(-\dfrac{b}{3a}, f\left(-\dfrac{b}{3a}\right)\right)$.

3. From $f'(x) = 3ax^2 + 2bx + c$, $f''(x) = 6ax + 2b = 0$ it follows that $x =$ $x = -\dfrac{b}{3a}$, and points $\left(-\dfrac{b}{3a}, f\left(-\dfrac{b}{3a}\right)\right)$ are the inflection points of $f(x)$.