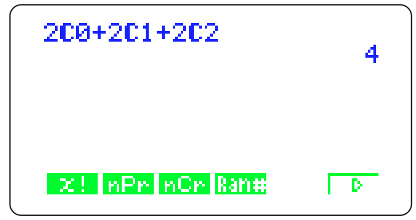
(3) Number of cases and probability

Problem 7: **Represent $_n\mathrm{C}_0 + {_n\mathrm{C}_1} + {_n\mathrm{C}_2} + {_n\mathrm{C}_3} + \cdots + {_n\mathrm{C}_n}$ in terms of n (n is a natural number).**

<Understanding the problem using a graphic calculator>

Select RUN from the MENU, and press [OPTN] [F6] (▷) [F3] (PROB). Then, input [2] [F3] ($_nC_r$) [0] [+] [2], [F3] ($_nC_r$) [1] [+] [2] [F3] ($_nC_r$) [2], and press [EXE] to display $_2C_0 + {_2C_1} + {_2C_2}$ and its value as shown in Figure 22. Input $_3C_0 + {_3C_1} + {_3C_2} + {_3C_3}$ and determine its value as before.
From $_1C_0 + {_1C_1} = 2$, $_2C_0 + {_2C_1} + {_2C_2} = 4$, $_3C_0 + {_3C_1} + {_3C_2} + {_3C_3} = 8, \cdots$, it may be inferred that the equation is 2^n. Based on that observation, continued input of $_nC_r(r = 0, 1, 2, \cdots, n)$ may yield a solution by considering the number of cases for number I.

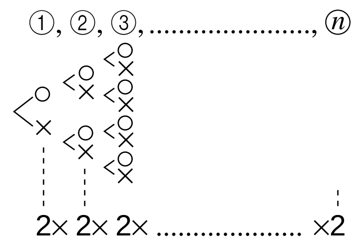


(Figure 22)

Solution

The equation determines r number of items yielded from a distinct number of n items. If r number of items is represented as r = 0, 1, 2,···, then the summation of $_nC_r$ will determine the value. If the distinct number of n items ((1), (2), (3),..., (n)) is assigned to each set yielded as shown in the diagram at right, then each value can be confirmed. Therefore, the total number is:

$$_nC_0 + {_nC_1} + {_nC_2} + \cdots {_nC_n} = 2 \times 2 \times 2 \times \cdots \times 2$$
$$= 2^n \text{ .... (solution)}$$



(Figure)

Comments

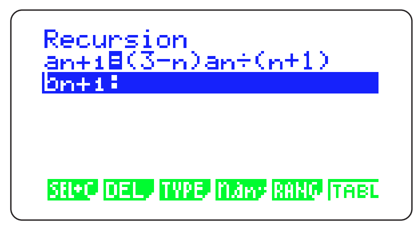
1. Of course, the solution is easily obtained by substituting $a = b = 1$ in the binomial theorem $(a+b)^n = \sum_{r=0}^{n} {_nC_r} a^r b^{n-r}$. However, the solution is also fully possible using a calculator.

2. The mathematically induced proof also uses $_{n+1}C_r = {_nC_r} + {_nC_{r-1}}$.

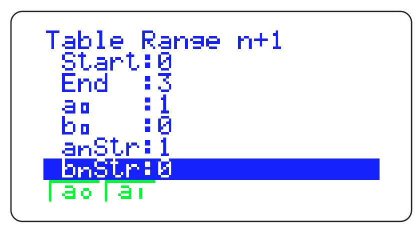
3. Use of $_nC_{r+1} = \dfrac{n-r}{r+1}\,{_nC_r}$ allows rapid calculation.

   For example, the equation becomes $a_{r+1} = \dfrac{3-r}{r+1} a_r$ when n = 3. Select RECUR from the MENU, and input as shown in Figure 23. Then, set the Table Range as shown in Figure 24. Press [SHIFT] [MENU] (SET UP) [F1] to set the Display: ON, and then press [EXE] twice to display the value of $_3C_0 + {_3C_1} + {_3C_2} + {_3C_3}$ in the lower right of the screen as shown in Figure 25.

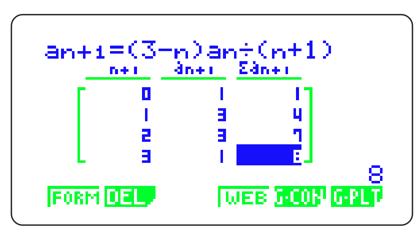
4. Determination of $\sum_{r=0}^{n} n \bullet {_nC_r}$ can also be considered for course development purposes.



(Figure 23)



(Figure 24)



(Figure 25)

Problem 8: **Give one card each to *n* number of students, and then have the students write a number from 1 to 10 on their cards. Determine the range of *n* required for the probability that at least two cards will have the same number to exceed 0.9.**

<Understanding the problem using a graphic calculator>

If the probability that n number of students will write different numbers is a_n and the remaining probability is b_n, then the equation can be expressed as:

$$a_n = \frac{{}_{10}P_n}{10n} (2 \leqq n \leqq 10),\ a_n = 0\ (n \geqq 11)$$

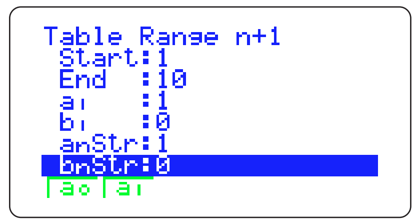
$$a_{n+1} = \frac{{}_{10}P_{n+1}}{10^{n+1}} = \frac{10-n}{10} a_n,\ b_n = 1 - a_n (2 \leqq n \leqq 10)$$

Therefore, a_1=1, b_1=0.

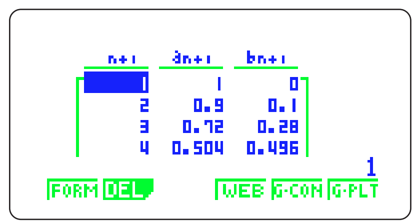
Select RECUR from the MENU, and input

$$a_{n+1} = (10-n)a_n \div 10,\ b_{n+1} = 1-(10-n)a_n \div 10.$$

Press [F4] (RANG) to set the Table Range as shown in Figure 26, and then press [SHIFT] [MENU] (SET UP) to set Sigma Display: OFF. Next, press [EXE] twice to obtain the value of a_{n+1}, b_{n+1} for each $n+1$ as shown in Figure 26. From the table shown in Figure 27, it is learned that a_n decreases and b_n increases as n increases, and that $b_n > 0.9$ when $n \geqq 7$. This means that $a_n + 1 < a_n$ when $2 \leqq n \leqq 10$, and the solution is obtained by showing that $a_7 < 0.1 < a_6$.



(Figure 26)



(Figure 27)

Solution

If the probability that n number of students will write different numbers is set to a_n, then $1 - a_n > 0.9$. One must determine the range of n that produces $a_n < 0.1$.

Next, set a_n = 1, and express the formula as $a_n = \frac{{}_{10}P_n}{10^n} (1 \leqq n \leqq 10),\ a_n = 0\ (n > 10)$

Therefore, from $a_{n+1} = \frac{{}_{10}P_{n+1}}{10^{n+1}} = \frac{10-n}{10} a_n$

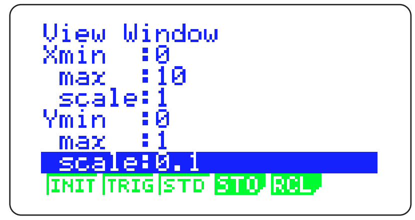
When $2 \leqq n \leqq 10$, $a_n + 1 < a_n$ from $0 < \frac{10-n}{10} < 1$ .... (1)

Additionally, $a_6 = 0.152 > 0.1 > a_7 = 0.0604$ .... (2)

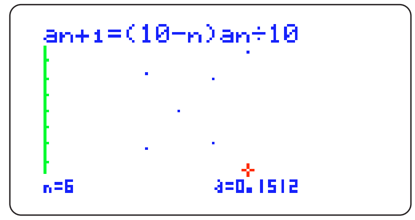
Therefore, from (1) and (2), $n \geqq 7$ .... (solution)

Comments

1. In this case, the problem is the increases and decreases of *an* and *bn*. Therefore, press [SHIFT] [F3] (V-WIN) and set the ranges of the *X* and *Y* axes as shown in Figure 28, and then press [F6] (TABL) [F6] (G-PLT) to plot the graph as shown in Figure 29. Press [SHIFT] [F1] (TRCE) and use the cursor (◁▷) to view the changes as desired.
2. Further development can be effected by using the "birthday problem." Change 10 to 365, and determine the probability that any two persons within *n* number of persons will have the same birthday.
3. Only the calculations for this problem can be done on a calculator, and prohibiting the use of a calculator causes considerable resistance because of the number of calculations.

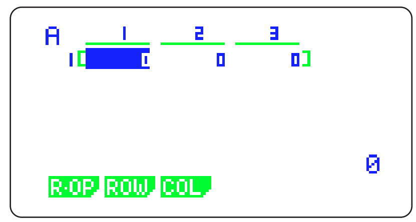


(Figure 28)



(Figure 29)

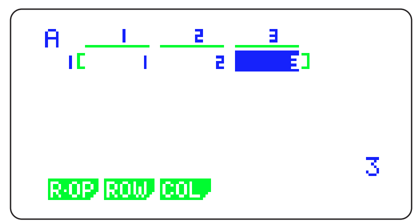
Problem 9: **From set $A = \{1, 2, 3, \cdots, n\}$, make f a map that has 1:1 correspondence to set A. What is the range of $\sum_{i=1}^{n} i \bullet a_i$ when $f(i) = a_i$ ($i = 1, 2, 3, \ldots, n$)? Express in terms of n.**



(Figure 30)

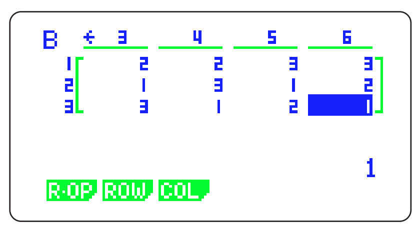
<Understanding the problem using a graphic calculator>

Select MAT (matrix) from the MENU, and input the dimensions (1 x 3) of matrix A to display a 1 x 3 matrix containing all zeros as shown in Figure 30. Next, press [1] [EXE] [2] [EXE] [3] [EXE] to input the components $A = (1, 2, 3)$ as shown in Figure 31.



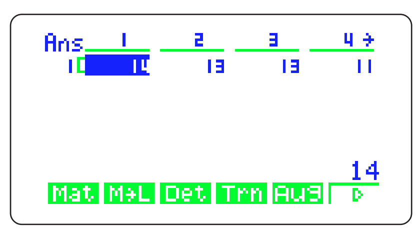
(Figure 31)

As before, make matrix *B* a 3 x 6 matrix as shown in Figure 32, and input the components so that each column has a different permutation of 1, 2, and 3. Then, press [MENU] (RUN), [OPTN] [F2] (MAT) [F1] (MAT) [ALPHA] [A] [X] [F1] (MAT) [ALPHA] [B] [EXE] to display the calculation results of $A \bullet B$ as a matrix as shown in Figure 33.
Next, input $C = (1, 2, 3, 4)$, and make *D* a 4 x 24 matrix. Input the components so that each column of *D* has a different permutation of 1, 2, 3, and 4. Then, calculate $C \bullet D$.



(Figure 32)

From these calculator operations, it can be surmised that $f(i) = i$ produces the maximum, and $f(i) = n+1-i\ (i = 1, 2, 3, \cdots, n)$ produces the minimum. From matrices *B* and *D* it can also be confirmed that map *g* produced by $g(i) = n + 1 - a_i$ must exist with respect to map *f* produced by $f(i) = a_i$. This fact may cause one to surmise that if the maximum occurs when $f(i) = a_i$, and that the minimum value may also occur when $f(i) = n+1-i$.



(Figure 33)

Solution

$(1^2+2^2+3^2+\cdots+n^2)\,(a_1{}^2+a_2{}^2+a_3{}^2,+\cdots+a_n{}^2)-(1\cdot a_1+2\cdot a_2+3\cdot a_3+\cdots+n\cdot a_n)^2$

In short,

$=(1\cdot a_2-2\cdot a_1)^2+(1\cdot a_3-3\cdot a_1)^2+\cdots+(1\cdot a_n-n\cdot a_1)^2$

$+(2\cdot a_3-3\cdot a_2)^2+\cdots+(2\cdot a_n-n\cdot a_2)^2$

$+\cdots+((n-1)\cdot a_n-n\cdot a_{n-1})^2$

$\geqq 0$

Therefore, $\left(\sum_{i=1}^{n} i^2\right)\cdot\left(\sum_{i=1}^{n} a_i{}^2\right)\geqq\left(\sum_{i=1}^{n} i\cdot a_i{}^2\right)^{2\,*}$

Then, $\sum_{i=1}^{n} a_i{}^2=\sum_{i=1}^{n} i^2=\dfrac{n(n+1)\,(2n+1)}{6}$

Thus, $\therefore\left(\sum_{i=1}^{n} i\cdot a_i\right)^2\leqq\left\{\dfrac{n(n+1)\,(2n+1)}{6}\right\}^2$

Additionally, $\sum_{i=1}^{n} i\cdot a_i>0,\ \dfrac{n(n+1)\,(2n+1)}{6}>0$

Therefore, $\therefore\sum_{i=1}^{n} i\cdot a_i\leqq\dfrac{n(n+1)\,(2n+1)}{6}$ .... (1)

(The equality occurs when $i\cdot a_j-j\cdot a_i=0\ (i,\ j:1,2,\cdots,n)$, in short, when $a_i=i$)

Substituting $b_i=n+1-a_i$ produces:

$$\sum_{i=1}^{n} i\cdot b_i=(\mathrm{n}+1)\sum_{i=1}^{n} i-\sum_{i=1}^{n} i\cdot a_i$$

$$\geqq\frac{n(n+1)^2}{2}-\frac{n(n+1)(2n+1)}{6}\quad(\because(1))$$

$$=\frac{n(n+1)(n+2)}{6}\ \text{.............................}\ (2)$$

allow students to increase their problem-solving abilities.