Thus, from the fact that $f$ from $A$ produces a 1:1 map on $A$ when $f(i) = a_i$, then map $g$ (produced by $g(i) = b_i = n+1-a_i$) must exist.

Therefore, from (1) and (2), the minimum value of $\sum_{i=1}^{n} i \cdot a_i$ occurs in (2) when $a_i = n + 1 - i$.

This produces:

$$\frac{n(n+1)(n+2)}{6} \leqq \sum_{i=1}^{n} i \cdot a_i \leqq \frac{n(n+1)(2n+1)}{6} \cdots \text{(solution)}$$

**Comments**

1. The proof of the inequality * can be simplified by using Schubert's inequality.

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geqq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$$

2. The proof of the inequality * becomes $\sum_{i=1}^{n} (pa_i + i)^2 \geqq 0$ by substituting $(pa_i + i)^2 \geqq 0$

($p$ is a non-zero real number).

Therefore, $\left(\sum_{i=1}^{n} a_i^2\right) p^2 + 2p\sum_{i=1}^{n} i \cdot a_i + \sum_{i=1}^{n} i^2 \geqq 0$

Here, it can be deduced from $\sum_{i=1}^{n} a_i^2 > 0$ that the discriminant must be $\geqq 0$.

Additionally, from $\sum_{i=1}^{n} (i - a_i)^2 \geqq 0$ the expansion is $2\sum_{i=1}^{n} i \cdot a_i \leqq \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} a_i^2$

$$= 2\sum_{i=1}^{n} i^2 \ (\because \sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} a_i^2)$$

Therefore, we can also say that $\sum_{i=1}^{n} i \cdot a_i \leqq \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

## (5) Locus and domain

**Problem 10:** $x = t + \frac{1}{t}, y = t^2 + \frac{1}{t^2} - 2t - \frac{1}{t}$

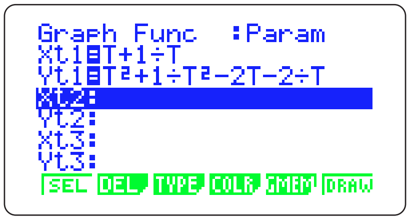
**Determine the locus of point $P$ ($x$, $y$) when $t$ changes ($t$ is a non-zero real number).**

### <Understanding the problem using a graphic calculator>

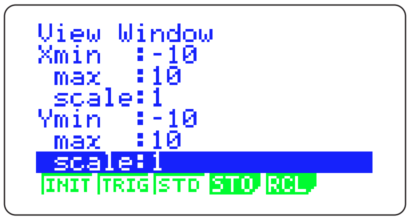
Select GRAPH from the MENU, and press F3 (TYPE) to select F3 (Parm • Parameter: Additional Variable Display). Then, input X,θ,T + 1 ÷ X,θ,T EXE X,θ,T $x^2$ + 1 ÷ X,θ,T $x^2$ − 2 X,θ,T − 2 ÷ X,θ,T to produce the graph functions shown in Figure 34.

Press SHIFT F3 (V-WIN) to set the range of the $X$ and $Y$ axes as shown in Figure 35. Then, press the cursor down key (⊙) until the screen shown in Figure 36 appears, and set the range of $T$, $\theta$ in the same View Window.

Next, press EXE twice to display the graph of point $P$ ($x$, $y$) as shown in Figure 37. Press SHIFT F1 (TRCE), and press the cursor right key (⊙) to move the pointer (+) to the endpoints. The $X$, $Y$ values can then be read from the display screen. (Figures 38 and 39)


Graph Func :Param
Xt1■T+1÷T
Yt1■T²+1÷T²−2T−2÷T
Xt2:
Yt2:
Xt3:
Yt3:

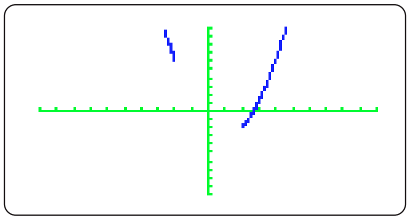
(Figure 34)


View Window
Xmin :-10
max :10
scale:1
Ymin :-10
max :10
scale:1

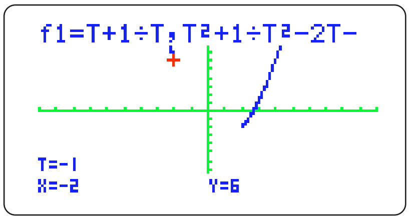
(Figure 35)


View Window
T,θ
min :-10
max :10
pitch:0.1

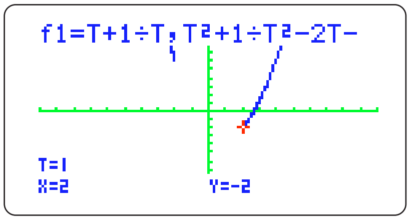
(Figure 36)



(Figure 37)


f1=T+1÷T,T²+1÷T²−2T−
T=-1
X=-2
Y=6

(Figure 38)


f1=T+1÷T,T²+1÷T²−2T−
T=1
X=2
Y=-2

(Figure 39)

**Solution**

From $y = \left(t + \frac{1}{t}\right)^2 - 2\left(t + \frac{1}{t}\right) - 2,\ x = t + \frac{1}{t}$

$$y = x^2 - 2x - 2$$

$$= (x-1)^2 - 3$$

Then, from $t \neq 0$, $|t| > 0$, so $\frac{1}{|t|} > 0$

Therefore, $\frac{|t| + \frac{1}{|t|}}{2} \geqq \sqrt{|t| \cdot \frac{1}{|t|}}$ $\quad \therefore |t| + \frac{1}{|t|} \geqq 2$

Here, $t$ and $\frac{1}{t}$ have the same sign, so $t + \frac{1}{t} = x \geqq 2$ when $t > 0$.

When $t < 0$, $-t - \frac{1}{t} = \geqq -2$ $\quad \therefore x = t + \frac{1}{t} \leqq -2$

Therefore, the locus of point $P$ ($x$, $y$) is:

$$y = (x-1)^2 - 3 \qquad (|t| > 2) \text{ ....... (Solution)}$$

## Comments

1. Setting the range of $t$ as shown in Figure 36 includes $t = 0$. However, when $t = 0$, this calculator instantaneously jumps to a non-zero value of $t$. This value is not mathematically possible. Pressing MENU (RUN) for $1 \div 0$ will cause a Ma ERROR because division by zero is impossible. This impossibility is the reason for the automatic jump to a non-zero value of $t$.

2. To determine the range of $x$ from a graph, select GRAPH from the MENU and select F3 (TYPE) F1 (Y–). Press SHIFT F5 (G-SLV) for the graph of $Y2 = X + 1 \div X$ and use the pointer (+) to display the minimum value when $x > 0$ on the screen.

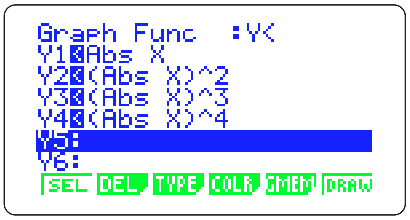
## Problem 11: $E_n$ is a set of points $(x, y)$ on a plane that satisfy the inequality $y < |x|^n$. Diagram the entire set of points $(x, y)$ common to $E_n$ ($n$ = 1, 2,...).

### <Understanding the problem using a graphic calculator>

Select GRAPH from the MENU, and press F3 (TYPE) F6 (▷) F2 (Y<). Then, press OPTN F5 (NUM) F1 (ABS: absolute value) and input $Y <$ Abs $X$.

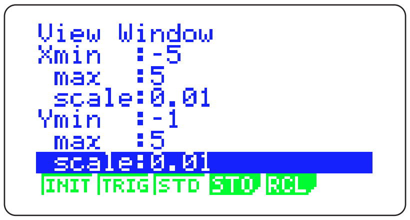
Input the inequalities in the same manner as shown in Figure 40. Next, press SHIFT F3 (V-WIN) and set the ranges of $X$ and $Y$ as shown in Figure 41.

Then, press EXE twice to display the domain of the points in orange that are common to $y < |x|^n$ $(n = 1, 2, 3, 4)$ as shown in Figure 42. Press SHIFT F1 (TRCE) and move the pointer (+) to the intersections of these four functions. Reading of the coordinates as shown in Figure 43 allows one to infer that the intersections are (–1, 0), (0, 0),and (0, 1). Thus, all of the graphs are symmetrical to the $Y$-axis. Therefore, the solution can be obtained by writing two inequalities $0 < |x| < 1, |x| > 1$ and testing the maximum and minimum relationships of $Y1$, $Y2$, $Y3$, and $Y4$.



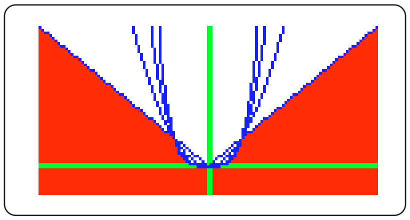
```
Graph Func  :Y<
Y1<Abs X
Y2<(Abs X)^2
Y3<(Abs X)^3
Y4<(Abs X)^4
Y5:
Y6:
```

(Figure 40)

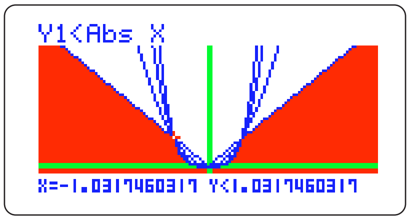


```
View Window
Xmin  :-5
 max  :5
 scale:0.01
Ymin  :-1
 max  :5
 scale:0.01
```

(Figure 41)



(Figure 42)



(Figure 43)

## Solution

Express the equation as $E_n = \{(x, y) | y < |x|^n\}$.

Multiplying both sides by $|x|(> 0)$ when $0 < |x| < 1$ yields:

$$0 < |x|^2 < |x| \text{ .................... (1)}$$

Multiplying both sides of (1) by $|x| > 0$ again yields:

$$0 < |x|^3 < |x|^2 \text{ .................. (2)}$$

Therefore, from (1) and (2):

$$0 < |x|^n < |x|^{n-1} < \cdots < |x|^2 < |x|$$

Thus, when $0 < |x| < 1$ the equation is $E_1 \cap E_2 \cdots \cap E_n = E_n$ ............ (3)

Handle the case of $|x| > 1$ in a similar manner:

$$|x| < |x|^2 < \cdots < |x|^n \text{ .......................... } (4)$$

This induction allows the following from (4) when $|x| > 1$.

$$E_1 \cap E_2 \cap \cdots \cap E_n = E_1 \text{ ..................... } (5)$$

Additionally, when $|x| = 0, 1$

$$|x| = |x|^2 = \cdots = |x|^n \text{ .......................... } (6)$$

Therefore, from (3), (5), and (6), the shaded area of the diagram represents the domain of the determined points (*x*, *y*). However, the dotted boundary lines and points are not included in the solution.

## Comments

1. The calculator operation displayed the domain of $y < |x|^n \ (n = 1, 2, 3, 4)$. However, a method that displays $y = |x|^n \ (n = 1, 2, 3, 4)$ again after the initial display of $y < |x|^n \ (n = 1, 2, 3, 4)$ may make it easier for the students to understand.

2. From this problem, diagramming of the domain of $E_1 \cup E_2 \cup \cdots \cup E_n$ also becomes easier.

## Concluding Remarks

The use of calculators to solve problems in Japanese college entrance examinations is currently not allowed. However, consideration of the widespread diffusion of computers in everyday life makes it clear that valuable mathematics time should not be spent on simple mechanical calculation techniques and practice. New educational perspectives that focus on students developing their study abilities suggest that we should provide an environment that permits spontaneous use of calculators. The intent is to allow students to increase their problem-solving abilities.