# Opening of Entrance Examination Problems Solutions Using a Graphic Calculator

## (1) Multiple Solutions to Inequalities

**Application: Quadratic functions**

**Objective:** **Functions and equations that use two or more characters often appear in college entrance examination problems. Distinguishing between variables, parameters, and constants causes trouble for students taking examinations. Use of graphic calculators allows students to freely change variables and observe the results.**

**Problem**

**Determine the range of the values of $a$ that satisfy the inequality $x < 3 + ax - x^2$ for all $x$ ($1 \leqq x \leqq 3$).**

**Solution**

Write the function as $f(x) = x2 + x - ax - 3$ limited by $1 \leqq x \leqq 3$. Determining the range of $a$ which causes $f(x) < 0$ is the gist of the problem. The $y$ coordinates of the vertex are < 0, and $f(1) < 0$, and $f(3) < 0$.

$$f(x) = x^2 + (1-a)x - 3 = \left(x + \frac{1-a}{2}\right)^2 - \frac{(1-a)^2}{4} - 3 \qquad \text{Vertex} \left(\frac{a-1}{2}, -\frac{(1-a)^2}{4} - 3\right)$$

$-\frac{(1-a)^2}{4} - 3 < 0$ Therefore, <u>all real $a$</u>

$f(1) = 1 + 1 - a - 3 < 0$ Therefore, <u>$a > -1$</u>

$f(3) = 9 + 3 - 3a - 3 < 0$ Therefore, <u>$a > 3$</u>

A. <u>$a > 3$</u>

### Solution 1 (using the graphic calculator)

**1** Use the dynamic graph function to make changes in coefficient $a$ of $f(x)$ and observe changes in the graph.

1. Select the dynamic function from the main menu.

2. Enter $x^2 + x - ax - 3$ as $f(x)$.
   Input $x^2 + x - ax - 3$ to $Y1$ and press [EXE] to enter the expression.

3. Then, press [EXE] to proceed to the next screen.

4. Next, set the values of $A$ from –4 to 4 in order to view changes in the graph.
   Press [F2] (RANG) to select the menu screen for setting of the variable range.
   Then, press [(–)] [4] [EXE] [4] [EXE] [1] [EXE] [EXE].

5. Then, press [F6] (DYNA).

**2** Use the overwrite graph function to make changes in coefficient $a$ of $f(x)$ and observe changes in the graph.

1. Select the graph mode from the main menu.
2. Enter $x^2 + x - ax - 3$ as $f(x)$.
   Input $x^2 + x - Ax - 3, [A = -4, -3, -2, -1, 0, 1, 2, 3, 4]$ to $Y1$ and press [EXE].
3. Press [F6] (DRAW).

Either the dynamic graph function or the overwrite function of the graphic calculator can be used to “confirm” the solution to this problem. However, it is impossible to determine the solution using only the calculator.

### Solution 2 (using the graphic calculator)

When reading the problem inequality $x < 3 + ax - x^2$, simply consider $a$ as a function of $x$. Then, graph $x = 3 + ax - x^2$ to determine the solution.

$$f(x) = \frac{x^2 + x - 3}{x} \qquad (x \neq 0)$$

Determine the range of $a$ that causes $a > f(x)$ using an arbitrary $x$ when $1 \leq x \leq 3$.

1. Select the graph mode from the main menu.
2. Define $\frac{x^2 + x - 3}{x}$ to be drawn within the range ($1 \leqq x \leqq 3$) by inputting $(x^2 + x - 3) \div x, [1, 3]$.
3. Press [EXE] to save the expression.
4. Press [F6] (DRAW).
5. Press [SHIFT] [F1] (Trace) and use the cursor right key (▶) to move the pointer (+) to the maximum value.

→ The maximum value occurs at $a = 3$ when $x = 3$.

A. <u>$a > 3$</u> (solution)

This problem can ultimately be solved with extensive use of the graphic calculator. Nevertheless, it must be confirmed in advance whether the graph represents a simply increasing function when the Trace function is used in step 5.

### Conclusion

The usage methods of the graphic calculator in this problem are:

a confirmation tool in Solution 1 (using the graphic calculator)
a tool for directly solving the problem in Solution 2 (using the graphic calculator)

**Problem Analysis:** **The range of $x$ $(1 \leqq x \leqq 3)$ L**

The following is the graph in Solution 2 (using the graphic calculator)

$$y = \frac{x^2 + x - 3}{x}$$

Adjusting the range of the graphic calculator lets you surmise that $x$ is a simply increasing variable. (This fact can also be confirmed by differentiating the right side of the function.)

$$\frac{d}{dx}\left(\frac{x^2 + x - 3}{x}\right) = \frac{x^2 + 3}{x^2} = 0$$

At this point, there is no asymptotic line and it is not really necessary to limit the range of $x$ to $1 \leqq x \leqq 3$.

However, when the value of $x$ is an integer, when does $\dfrac{x^2 + x - 3}{x}$ produce an integer value?

Use the TABLE function of the CFX-9850G to view the incremental changes in $y$ for values of $x$ from –100 to 100.

| X | Y |
|---|---|
| -100 | -98.97 |
| -99 | -97.96 |
| -98 | -96.96 |
| -97 | -95.96 |
| -96 | -94.96 |
| -95 | -93.96 |
| -94 | -92.96 |
| -93 | -91.96 |
| -92 | -90.96 |
| -91 | -89.96 |
| -90 | -88.96 |
| -89 | -87.96 |
| -88 | -86.96 |
| -87 | -85.96 |
| -86 | -84.96 |
| -85 | -83.96 |
| -84 | -82.96 |
| -83 | -81.96 |
| -82 | -80.96 |
| -81 | -79.96 |
| -80 | -78.96 |
| -79 | -77.96 |
| -78 | -76.96 |
| -77 | -75.96 |
| -76 | -74.96 |
| -75 | -73.96 |
| -74 | -72.95 |
| -73 | -71.95 |
| -72 | -70.95 |
| -71 | -69.95 |
| -70 | -68.95 |
| -69 | -67.95 |
| -68 | -66.95 |
| -67 | -65.95 |
| -66 | -64.95 |
| -65 | -63.95 |
| -64 | -62.95 |
| -63 | -61.95 |
| -62 | -60.95 |
| -61 | -59.95 |

| X | Y |
|---|---|
| -60 | -58.95 |
| -59 | -57.94 |
| -58 | -56.94 |
| -57 | -55.94 |
| -56 | -54.94 |
| -55 | -53.94 |
| -54 | -52.94 |
| -53 | -51.94 |
| -52 | -50.94 |
| -51 | -49.94 |
| -50 | -48.94 |
| -49 | -47.93 |
| -48 | -46.93 |
| -47 | -45.93 |
| -46 | -44.93 |
| -45 | -43.93 |
| -44 | -42.93 |
| -43 | -41.93 |
| -42 | -40.92 |
| -41 | -39.92 |
| -40 | -38.92 |
| -39 | -37.92 |
| -38 | -36.92 |
| -37 | -35.91 |
| -36 | -34.91 |
| -35 | -33.91 |
| -34 | -32.91 |
| -33 | -31.9 |
| -32 | -30.9 |
| -31 | -29.9 |
| -30 | -28.9 |
| -29 | -27.89 |
| -28 | -26.89 |
| -27 | -25.88 |
| -26 | -24.88 |
| -25 | -23.88 |
| -24 | -22.87 |
| -23 | -21.86 |
| -22 | -20.86 |
| -21 | -19.85 |

| X | Y |
|---|---|
| -20 | -18.85 |
| -19 | -17.84 |
| -18 | -16.83 |
| -17 | -15.82 |
| -16 | -14.81 |
| -15 | -13.8 |
| -14 | -12.78 |
| -13 | -11.76 |
| -12 | -10.75 |
| -11 | -9.727 |
| -10 | -8.7 |
| -9 | -7.666 |
| -8 | -6.625 |
| -7 | -5.571 |
| -6 | -4.5 |
| -5 | -3.4 |
| -4 | -2.25 |
| -3 | -1 |
| -2 | 0.5 |
| -1 | 3 |
| 1 | -1 |
| 2 | 1.5 |
| 3 | 3 |
| 4 | 4.25 |
| 5 | 5.4 |
| 6 | 6.5 |
| 7 | 7.5714 |
| 8 | 8.625 |
| 9 | 9.6666 |
| 10 | 10.7 |
| 11 | 11.727 |
| 12 | 12.75 |
| 13 | 13.769 |
| 14 | 14.785 |
| 15 | 15.8 |
| 16 | 16.812 |
| 17 | 17.823 |
| 18 | 18.833 |
| 19 | 19.842 |
| 20 | 20.85 |

| X | Y |
|---|---|
| 21 | 21.857 |
| 22 | 22.863 |
| 23 | 23.869 |
| 24 | 24.875 |
| 25 | 25.88 |
| 26 | 26.884 |
| 27 | 27.888 |
| 28 | 28.892 |
| 29 | 29.896 |
| 30 | 30.9 |
| 31 | 31.903 |
| 32 | 32.906 |
| 33 | 33.909 |
| 34 | 34.911 |
| 35 | 35.914 |
| 36 | 36.916 |
| 37 | 37.918 |
| 38 | 38.921 |
| 39 | 39.923 |
| 40 | 40.925 |
| 41 | 41.926 |
| 42 | 42.928 |
| 43 | 43.93 |
| 44 | 44.931 |
| 45 | 45.933 |
| 46 | 46.934 |
| 47 | 47.936 |
| 48 | 48.937 |
| 49 | 49.938 |
| 50 | 50.94 |
| 51 | 51.941 |
| 52 | 52.942 |
| 53 | 53.943 |
| 54 | 54.944 |
| 55 | 55.945 |
| 56 | 56.946 |
| 57 | 57.947 |
| 58 | 58.948 |
| 59 | 59.949 |
| 60 | 60.95 |

| X | Y |
|---|---|
| 61 | 61.95 |
| 62 | 62.951 |
| 63 | 63.952 |
| 64 | 64.953 |
| 65 | 65.953 |
| 66 | 66.944 |
| 67 | 67.955 |
| 68 | 68.955 |
| 69 | 69.956 |
| 70 | 70.957 |
| 71 | 71.957 |
| 72 | 72.958 |
| 73 | 73.958 |
| 74 | 74.959 |
| 75 | 75.96 |
| 76 | 76.96 |
| 77 | 77.961 |
| 78 | 78.961 |
| 79 | 79.962 |
| 80 | 80.962 |
| 81 | 81.962 |
| 82 | 82.963 |
| 83 | 83.963 |
| 84 | 84.964 |
| 85 | 85.964 |
| 86 | 86.965 |
| 87 | 87.965 |
| 88 | 88.965 |
| 89 | 89.966 |
| 90 | 90.966 |
| 91 | 91.967 |
| 92 | 92.967 |
| 93 | 93.967 |
| 94 | 94.968 |
| 95 | 95.968 |
| 96 | 96.968 |
| 97 | 97.969 |
| 98 | 98.969 |
| 99 | 99.969 |
| 100 | 100.97 |

This fact is also clear from $\frac{x^2+x-3}{x} = x+1+\frac{3}{x}$.

As learned from the table on the previous page only $x$ = -3, -1, 1, and 3 produce integer values for $y$.

It is believed that the creator of the problem deliberately chose a range of $x$ that would produce integer values of y. (This is only conjecture on my part.) Nevertheless, this does not change the fact the range considered was based on the assumption that students should use only pencils and erasers to solve problems. This consideration of the problem creator is not necessary if graphic calculators can be used. The scope of the problem can be increased to any range of $x$, including the impossibility of x = 0, if graphic calculators can be used.

### Solution 3 (using the graphic calculator)

It is clear from the above analysis that it is not necessary to limit the range of $x$ in the problem if graphic calculators are used. However, in Solution 2 (using the graphic calculator), the new problem becomes “division by $x$” when $f(x)$ is created. The range of x was specifically limited in Problem 86, so it really was not much of a problem.
Let’s expand the problem by removing the limited range of $x$. Then, a solution cannot be reached simply by using the graphic calculator. Naturally, the sign of the inequality reverses when $x < 0$.

Therefore,

$$f(x) = \frac{x^2+x-3}{x} \qquad (x \neq 0)$$

Thus, the range $a$ must produce the following conditions.

When $x > 0$,

$$a > f(x)$$

When $x < 0$,

$$a < f(x)$$

This becomes the solution. The method of using of the graphic calculator is similar to the method used in Solution 2 (using the graphic calculator), so it has been omitted. This could be a way to allow the use of graphic calculators to solve problems on college entrance examinations.