Opening of Entrance Examination Problems Solutions Using a Graphic Calculator

(2) Application of Inverse Logic

Application: Quadratic equations and functions

Objective: When solving quadratic equations that include characters, ignoring x and viewing the graphs of characters other than x allows a broader perspective of problems.

Problem

a. Determine the range of a (real numbers) for the quadratic equation $x^2 + x + a = 0$.

b. Determine the range of a (real numbers) that satisfies both quadratic equations $x^2 + x + a = 0$, $x^2 + ax + 1 = 0$.

c. Also, determine the range of a (imaginary numbers) that satisfies both quadratic equations.

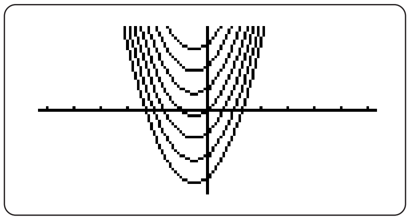
Solution 1 $y = x^2 + x + A, [A = -3, -2, -1, 0, 1, 2, 3]$

Notice that the x coordinates of the intersections of $y = x^2 + x + a$ and the x-axis are the real solutions of $x^2 + x + a = 0$. Use the overwrite function of the graphic calculator to determine the range of a that produces intersections of $y = x^2 + x + a$ and the x-axis.

Input the following to the overwrite function of the graphic calculator.

$$y = x^2 + x + A, [A = -3, -2, -1, 0, 1, 2, 3]$$

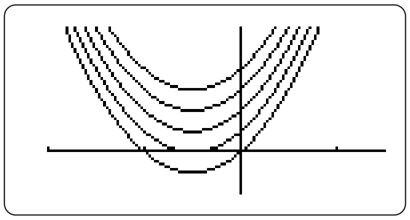
Note: The values of A may be arbitrarily set.



Then, changes in the number of intersections between $A = 0$ and $A = 1$ can be observed.

Next, set the values of A as follows.

$$y = x^2 + x + A, [A = 0, 0.25, 0.5, 0.75, 1]$$



Thus, the following inferences can be drawn.

There are two intersections when $A = 0$

There is tangency at a single point when $A = 0.25$

There are no intersections when $A = 0.5, 0.75, 1$

Thus, the following results are expected.

Two intersections when $A < 0.25$
Tangency at a single point when $A = 0.25$
No intersection when $A > 0.25$

Confirm it for yourself.

$$y = x^2 + x + a$$

Complete the square and determine the vertex.

$$y = \left(x + \frac{1}{2}\right)^2 + a - \frac{1}{4} \qquad \text{Vertex } \left(-\frac{1}{2}, a - \frac{1}{4}\right)$$

The essential condition for this function to intersect the x-axis is:

y-coordinates of the vertex $a - \frac{1}{4} \leqq 0$

$$a \leqq \frac{1}{4}$$

$$a \leqq 0.25$$

Thus, the expectation has been confirmed.

Then, the following conclusions can be made.
$x^2 + x + a = 0$ has:

Two real solutions when $a < 0.25$
multiple solutions when $a = 0.25$
an imaginary solution when $a > 0.25$

Consider $x^2 + ax + 1 = 0$ in the same manner, analyze the results, and the solution to the problem can be obtained. In the past, one had to begin by calculating by hand. Viewing the graph first and then determining the range is an application of the graphic calculator for inverse logic. It took considerable time and effort in the past, and the expectations and the solution did not always match. This application can be used as a way to deepen interest in mathematics.

Solution 2

Solve the given quadratic equations $x^2 + x + a = 0$, $x^2 + ax + 1 = 0$ for *a*.

$$-x^2 - x = a, -\frac{x^2 + 1}{x} = a \qquad (x \neq 0)$$

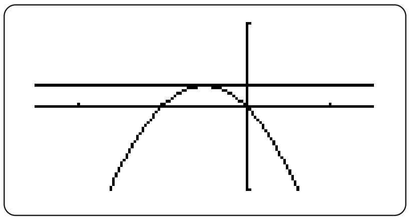
Then, analyze the graphs of these functions on the graphic calculator.

$$y = -x^2 - x, \; y = -\frac{x^2 + 1}{x}, \; y = a$$

Use the graph function to obtain the graph of $y = -x^2 - x$.
Use the Trace function to obtain the *y* coordinate.

$y = 0.25$

Overwrite the graph of $y = 0.25$



Then, the following analysis is possible.

The graph does not change when $a > 0.25$
The graph achieves tangency when $a = 0.25$
The graph intersects at two different points when $a < 0.25$

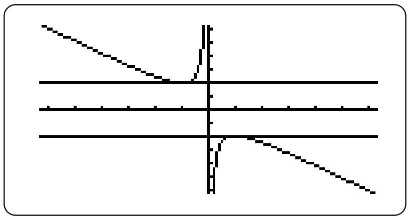
Therefore, $x^2 + x + a = 0$ has:

an imaginary solution when $a > 0.25$
a single real solution when $a = 0.25$
Various real solutions when $a < 0.25$

Use the graph function to obtain the graph of $y = -\frac{x^2+1}{x}$.
Use the Trace function to obtain the *y* coordinates.

$y_A = 2 \quad y_B = -2$

Overwrite the graphs of $y = 2$ and $y = -2$.



Then, the following analysis is possible.

The graph does not change when $-2 < a < 2$
The graph achieves tangency when $a = -2$, $a = 2$
The graph intersects at two different points when $a < -2$, $2 < a$

Therefore, $x^2 + ax + 1 = 0$ has:

an imaginary solution when $-2 < a < 2$
a single real solution when $a = -2$, $a = 2$
various real solutions when $a < -2$, $2 < a$