

CLEMSON ALGEBRA PROJECT UNIT 11: QUADRATIC MODELS

PROBLEM 1: JAR ROLL

Set a table on an angle. Take a cylindrical can or jar and push it so it rolls up and then back down the table. You are to investigate the object's position on the table over time.

- A. Determine and calculate an appropriate model.
- B. Investigate the errors in the model.
- C. Explore the differences in successive distance measurements. Make a conjecture as to what these differences represent, and model them mathematically.
- D. Finally explore the second order differences. Make conjectures as to what these differences, and model them mathematically.

MATERIALS

Casio CFX-9850Ga PLUS or ALGEBRA FX2.0 Graphing Calculator

QV780 Digital Camera and Meter Stick

Alternate to QV780 and Meter Stick: EA-100 Data Collector and Pasco Motion Detector

A cylindrical jar or can

EXTENSIONS

1. How does the model change if you move the motion detector or meter stick up or down a few centimeters?
2. What happens if you put the motion detector on the top of the table or reverse the meter stick, putting 0 at the top instead of the bottom?
3. What happens as you change the angle of the table?
4. Investigate the changes that occur if you roll different objects.
5. Investigate the changes that occur as you change the force with which you push the object up the incline.

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ONE SOLUTION TO PROBLEM 1: JAR ROLL

A. Determine and calculate an appropriate model.

1) Make a conjecture as to which model is appropriate. Discuss the pros and cons.

Due to gravity, the object will accelerate in the downward direction. Because the rate at which the object picks up speed is, at least theoretically, constant, the appropriate model is quadratic.

Students, of course, may have difficulty determining an appropriate model and may wish to look at a scatterplot before making a conjecture as to the type of mathematical model they should use. Although this is certainly a natural, intuitive approach, caution them that if they do not have a rationale for a particular model, they should conduct several experiments with as many data points as possible to verify that their model is appropriate. Without obtaining this confirmation, the model they have chosen may fit their particular sample data, but may not generalize to other similar situations.

Similarly, even though their calculators can perform very complex regressions, such as quartic, higher degree models require both a theoretical justification and an abundance of data points to have any validity. To convince yourself of this, make up any data for, say, six points. Calculate the quartic regression model and observe how well it fits your data. With only a few data points, complex models are too sample specific to be of much use in the real world. Simply put, they should not be generalized.

2) Gather the data with the QV-780 Digital Camera.

Prop up one side of a table or board that is at least a meter long. Attach a measuring tape or secure a meter stick to the table, with the 0 measure near the bottom. You will make a movie of the rolling object, so find a position above the table that allows you to see the entire meter stick. It will probably be necessary to make large marks near the meter stick so you can interpolate the measures more easily.

(NOTE: The data can be collected perhaps more easily with a motion detector, but giving students many opportunities to read and interpolate measuring

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devices is also worthwhile. Also getting the data collector to pick up only the jar can be problematic. As an alternate to the problem discussed, you may wish to use the data collector to chart the path of a free-falling object. The mathematics is fundamentally the same, but students will not see both “sides” of the parabola.)

Practice giving the can or jar a shove so that you release it near the bottom of the measuring device, and it rolls up and back down the table, taking approximately two seconds to complete its roll. Try different objects to find one that rolls smoothly. You will probably also have to experiment with the angle of the table.

- x Turn your digital camera on in RECO mode.
- x Press MODE a sufficient number of times until you see the movie icon. Find a position so that you can see the entire roll.
- x Working with your partner, make a movie of one complete roll by pressing and holding down the shutter. This may take a few attempts.
- x Switch your camera to PLAY mode, and find the movie. Press the shutter at the start of the movie, then MENU to pause the action. Use the + key to advance the movie one frame at a time.
- x Label any beginning point you observe during the roll 0. Set up a list of ordered pairs with x representing the frame number (beginning with 0) and y representing the measure on the meter stick. Construct the lists by advancing through the movie one frame at a time. Because the camera takes a picture every tenth of a second, count by .1 to build your x list, and use the observed location to build List 2. Be consistent with the part of the object you are looking for. Construct your lists for the duration of the roll. If the roll lasted two seconds, you will have 21 data points to use.
- x We now need to enter the data into the calculator. Turn your calculator on. From the MAIN MENU, call up the “List” mode.

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- x If there are data in Lists 1 and 2, press **F4** followed by **F1** to delete all of the elements. Alternatively, press **SHIFT** **MENU** for the setup and press a function key to work in a different List File.
- x In List 1, enter the times. For the experiment conducted in the development of the unit, the 17 data points are shown on the four screens below. Note that points 14 and 15 are shown twice.

	List 1	List 2	List 3	List 4
1	0	38		
2	0.1	50		
3	0.2	60		
4	0.3	69		
5	0.4	79		

SRTA
SRTD
DEL
DELQ
INS

	List 1	List 2	List 3	List 4
6	0.5	82		
7	0.6	84		
8	0.7	85		
9	0.8	84		
10	0.9	82		

SRTA
SRTD
DEL
DELQ
INS

	List 1	List 2	List 3	List 4
11	1.0	79		
12	1.1	74		
13	1.2	66		
14	1.3	59		
15	1.4	49		

SRTA
SRTD
DEL
DELQ
INS

	List 1	List 2	List 3	List 4
14	1.3	59		
15	1.4	49		
16	1.5	38		
17	1.6	22		
18				

SRTA
SRTD
DEL
DELQ
INS

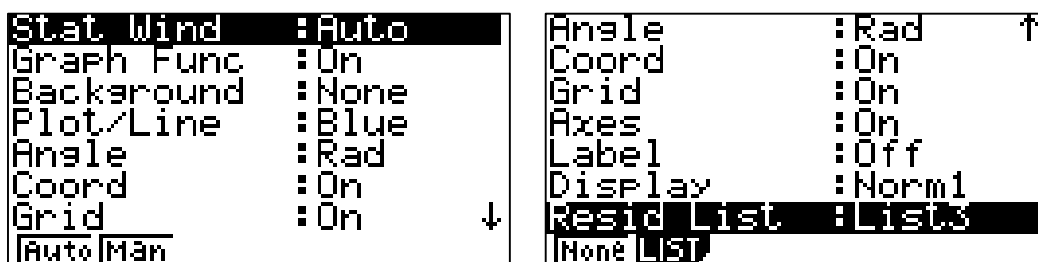
3) Construct a scatterplot, setting an appropriate window.

Before constructing the scatterplot, we'll make sure the window will be set automatically and that a list of residuals will be created automatically for us.

- x From the MAIN MENU, choose "Statistics."
- x Press **SHIFT** **MENU** to access the SET UP.
- x Make sure the "Stat Window" is set on "Automatic." If necessary, with "Stat Window" highlighted, press **F1**. See below left.

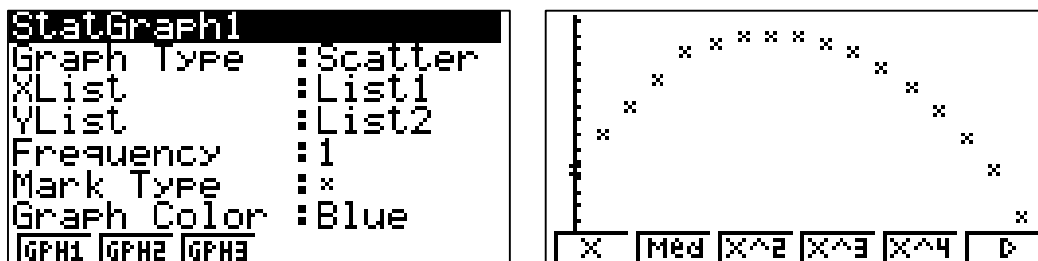
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- x Use the down arrow to highlight “Resid List.”
- x Press **F2** for LIST.
- x Press **F3** to have residuals created automatically in List 3. See below right.
- x Press **EXIT** to return to the primary statistics screen.



We're now ready to create the scatter plot.

- x Press **F1** to go to the GRPH menu, then **F6** for SET.
- x Set StatGraph1 as a Scatter graph, using List1 for the Xlist and List2 for the Ylist. The Frequency should be 1, and the Mark Type and Graph Color can be whatever you desire. See below left.
- x When finished, press **EXIT** to return to the graph menu screen.
- x Press **F1** to see StatGraph1, the scatterplot. See below right.

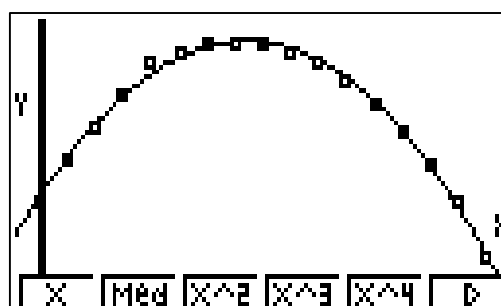
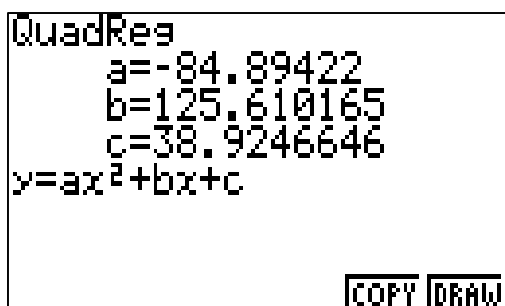


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Your graph should look somewhat similar to the one above. By pressing **SHIFT** **F1** , you can access the Trace function and use the right and left arrows to see the values of each of the data points. Pressing **EXIT** and **F1** will return you the graph without the Trace function.

4) Calculate the regression model and graph the regression model on the scatterplot.

- x With the scatterplot in view, the bottom menu gives several regression models. Because we are interested in a quadratic model, press **F3** for the X^2 , or second-degree regression. See the screen below left.
- x Press **F5** and **EXE** to copy the regression equation into Y1. Press **F6** to draw the regression equation on your scatterplot, giving a visual representation of how well the mathematical model fits the data. This is shown in the figure on the right below.



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5) *Interpret the y-intercept and other features of the regression equation.*

Rounded to four significant digits, the regression equation is $y = -84.89x^2 + 125.6x + 38.92$. The y-intercept, the value obtained for y when x is 0, is 38.92. Too often in using models, students do not make sense of the variables, of putting real-world meaning to the variables. In the model here, x represents time, measured in seconds. (When making a movie, the camera snaps a picture every tenth of a second.) The y-value represents the distance the top of the jar was from the bottom of the meter stick for any particular x-value. The starting point, the meaning of 0, was arbitrary. It was used at the first frame in the movie for which the object was rolling freely. According to the model, at time 0 the object should have been 38.92 centimeters from the bottom of the meter stick. Compare this with the data point (0, 38) used to construct the model. The error, or residual, is the distance between the y-value obtained and the y-value predicted by the model. For this point, the error is $38 - 38.92$, or $-.92$ centimeters. In other words, the point on the scatterplot is .92 centimeters below the point on the graph.

Also notice the negative value for a in the regression model. This indicates that the parabola opens downward. This should become intuitive for students, as gravity constantly pulls downward on the object. Even though we roll the object up the plane, the force acting on the object increases its speed in a downward direction. The object will reach a maximum height, and then move downward.

B. Investigate the errors in the model.

6) *Calculate the residuals, and make a scatterplot of them to determine if the model is appropriate. Make conjectures as to underlying reasons for the residuals.*

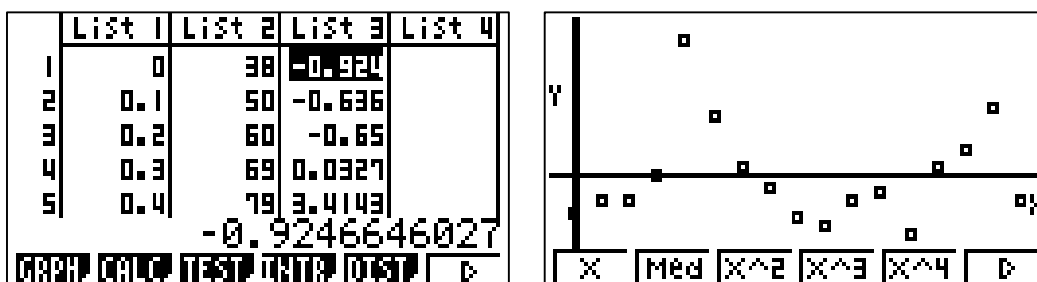
The errors, or residuals, in the model are the differences between the observed y-values (the position on the meter stick) and the y-values predicted by the regression equation. Because of what we did earlier in the SET UP for the Statistics, the residuals were automatically calculated and stored for us in List 3. The screen shown below left shows the beginning of this residual list.

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A scatterplot of these residuals is often valuable. Unlike much in the world of mathematics, our hope is that there is no pattern to the residuals! If there is one, then our model is not the most appropriate for the data. Below right shows the scatterplot of residuals, using List 1 as the x -value and List 3 as the y -value. To obtain the graph, starting at the STAT menu:

- x Press **F1** for the GRPH menu.
- x Press **F6** to SET the graph.
- x Press **F2** to work on GPH2, making the appropriate choices for the scatterplot. Press **EXIT** when finished.
- x Finally, press **F2** to see the graph.

The residual plot, while perhaps not as chaotic as we might wish, does not show a clearly discernible pattern for our data points.



Many causes for errors in our regression model are likely. First and foremost, reading the measures from the camera required interpolation and some guesswork. Also, the object may not have been perfectly cylindrical, causing disruptions in its journey. Similarly, inconsistencies in the density of the object may have caused it to wobble. The table itself may not have been smooth. The angle of the roll may not have been proper, causing the object to twist somewhat on its path. The camera may have been somewhat inconsistent in shooting consecutive frames. Considering all of these likely sources of error, our model is remarkably good! Nevertheless, students who are accustomed to thinking of mathematics as unambiguous and even perfect may have difficulty accepting the idea that a mathematical model can be flawed.

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7) *Calculate the mean of the residuals, the sum of the residuals, and the sum of the squared residuals.*

- x From the STAT menu, choose **F2** for CALC.
- x Press **F6** to SET the calculations up and **F3** so that one-variable statistics are performed on List 3.
- x Then press **EXIT** to back up to the "CALC" screen, followed by **F1** to perform the one-variable calculations. The screen below shows the results.

```
1-Variable
x̄ = 1.647E-13
Σx = 2.8E-12
Σx² = 24.4422084
x̄n = 1.19907337
x̄n-1 = 1.23597654
n = 17
[1VAR] [2VAR] [REG] [SET]
```

Note that the mean of the residuals, denoted by \bar{x} , is extremely close to 0 as is the sum of the residuals, denoted by $\sum x$. Students new to regression may need some explanation as to why they should interpret the values shown on the calculator as 0. This is an important idea when using technology; calculators must use a finite number of significant digits when making calculations, so some rounding error often occurs. Students should also recognize that the average deviation and sum of deviations would be 0 for several models (e.g., a least-squares linear regression model, even a model that uses the average value of y as a predictor no matter the value of x). Consequently, using the sum of residuals to determine how well a model fits a particular data set is not very helpful. To avoid this, which is caused by the cancellation of positive errors by negative errors, the sum of squared errors is often used to compare mathematical models.

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C. Explore the differences in successive distance measurements. Make a conjecture as to what these differences represent, and model them mathematically.

8) Create and investigate a list showing consecutive differences in the distances.

We will now create a list of successive differences to study, storing it in List 4. To do so:

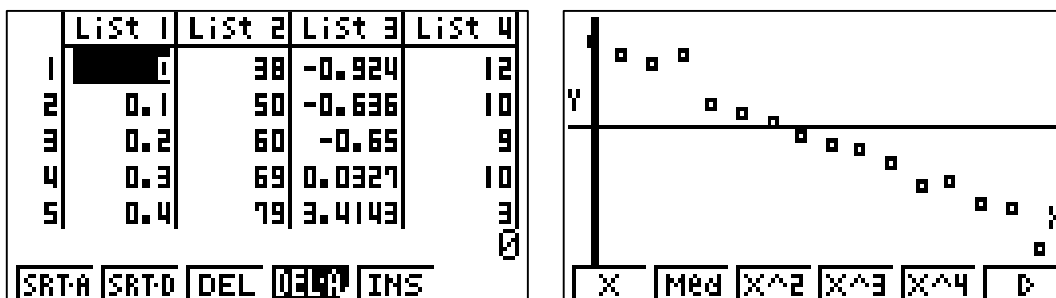
- x From the MAIN MENU, choose “Run.”
- x Press $\boxed{\text{OPTN}}$, $\boxed{\text{F1}}$ for LIST, $\boxed{\text{F6}}$ for more, and $\boxed{\text{F6}}$ for more again.
- x Now choose $\boxed{\text{F5}}$ for Δ List, 2 for List 2, $\boxed{\rightarrow}$ for store, $\boxed{\text{F6}}$ for more, $\boxed{\text{F1}}$ for List, $\boxed{4}$ for List 4, and finally $\boxed{\text{EXE}}$ to perform the function.

The calculator should say “Done,” and if you go to the STAT menu, you should see the screen on the below left.

We are now interested in looking at a scatterplot of these values, using List 1 for the x -values and our new List 4 for the y -values. To set up the scatterplot:

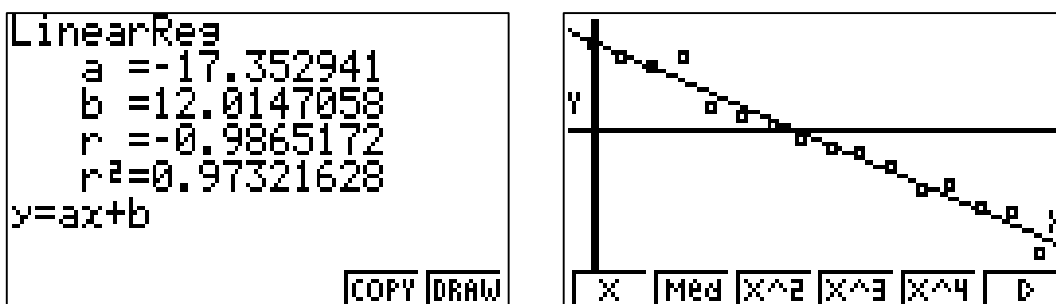
- x Press $\boxed{\text{F1}}$ for GRPH.
- x Press $\boxed{\text{F6}}$ for SET and $\boxed{\text{F3}}$ for StatGraph 3. Then make sure you set up a SCATTER graph, using List 1 as the Xlist and List 4 for the Ylist. Press $\boxed{\text{EXIT}}$ to back up a menu.
- x Before you can display the graph, one more thing must be done. Since we are looking at differences, there is one less value in List 4 than there is in List 1. We need to delete the last value from List 1. To do so, highlight the last value, press $\boxed{\text{EXIT}}$ to return to the primary “Statistics” screen and $\boxed{\text{F6}}$ for more options. Scroll down to the bottom of List 1, and press $\boxed{\text{F3}}$ to delete the last value.
- x Then press $\boxed{\text{F6}}$ for more options, $\boxed{\text{F1}}$ for GRPH, and $\boxed{\text{F3}}$ for GPH3. The results are shown below right.

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9) *Interpret these differences and explore an appropriate regression model.*

The scatterplot shows a reasonably consistent downward trend. Due to the consistency in the trend, a linear model may be most appropriate. In interpreting this, we should make use of the labels attached to the variables. The y-values show the distances traveled over successive time periods of .1 seconds. In other words, our y-values now measure velocity in centimeters per second. Positives reflect upward movement, and negatives reflect downward movement. The velocities should decrease, because gravity consistently pulls downward on the object. What we are actually seeing is that the velocity, the change in distance over time, is consistently decreasing. If we perform linear regression by pressing **F1** on the scatterplot, we see the screen below left. The graph of this, obtained by pressing **F6** for DRAW, is shown below right.



Note the high correlation of this linear model, indicating a good fit for the data. The y-intercept of 12.01 indicates that initially the jar's distance was changing at 12.01 centimeters per second. The positive value indicates that the object was moving upward. The slope, -17.35 , shows that, on average, for every second, the velocity was decreasing at a rate of -17.35 centimeters per second.

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D. Finally explore the second order differences. Make a conjecture as to what these differences mean, and model them mathematically.

10) Create and investigate a list showing consecutive differences in the velocities.

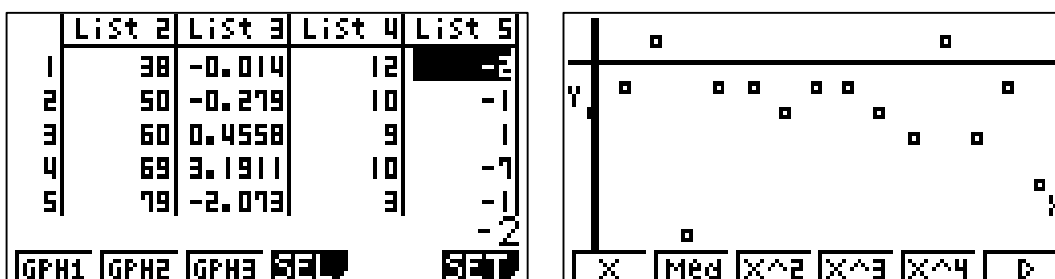
We will now investigate how much these velocities are changing. Just as we created a list showing the differences in the original distances (List 2), we will now create a new list, List 5, which will show the changes in velocity over time. The paragraphs below repeat the procedures we completed above, except that now we are investigating the differences in List 4. We will again use GRPH 3 for our scatterplot. To do so, from the MAIN MENU, choose “Run.”

- x Press **OPTN** , **F1** for LIST, **F6** for more, and **F6** for more again.
- x Press **F5** for Δ List, 4 for List 4, **→** for store, **F6** for more, **F1** for List, **5** for List 5, and finally **EXE** to perform the function. The calculator should say “Done,” and if you go to the STAT menu, you should see the screen on the below left.

Again, we wish to look at a scatterplot of these values, using List 1 as the x -value and our new List 5 as the y -values. To set up the scatterplot:

- x Press **F1** for GRPH, **F6** for SET, and **F3** for StatGraph3. Then make sure you set up a SCATTER graph, using List 1 as the Xlist and List 5 for the Ylist. Press **EXIT** to back up a menu.
- x Again, before you can display the graph, one more thing must be done. Since we are looking at differences, there is one less value in List 5 than there is in List 1. We need to delete the last value from List 1. To do so, highlight the last value, press **F6** for more options, and press **F3** to delete the last value.
- x To return to the graph, press **F6** for more options, **F1** for GRPH, and **F3** for GPH3. The results are shown below right.

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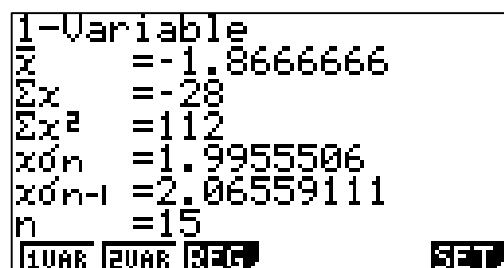
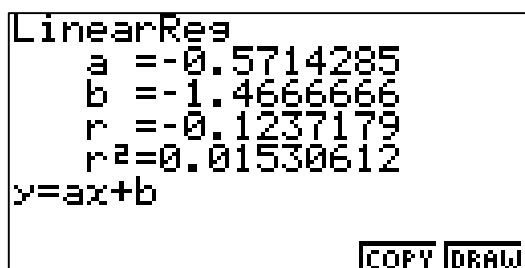


The graph seems to hang consistently a little below the x -axis. In other words, the velocity is changing at a negative rate, something that should not be very surprising since gravity was pulling downward on the object throughout its roll.

11) *Explore and interpret an appropriate model for these differences.*

This time, linear regression (obtained by pressing **F1** on the scatterplot) produces a very low correlation. (See below left.) In other words, the change in velocity is not dependent on the velocity it is rolling at any particular time. If we perform one variable statistics on List 5, the calculator will determine the average change in velocity for us. To do so, from the screen below left,

- x Press **EXIT** twice to back up to the primary “Statistics” screen and **F2** for “CALCULATIONS.”
- x Press **F6** for SET and **F5** so statistics are calculated on List 5.
- x Press **EXIT** . Press **F1** for one-variable statistics. See below right.



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The mean value of -1.867 tells us that, on average, each second the velocity was slowing down at the rate of 1.86 centimeters per second, or, as students may have heard, 1.86 centimeters per second squared.

An important point for students to explore here is that, although their scatterplots resemble lines, constant functions are not linear functions. This is because, for constant functions, the value of y does not depend on the value of x . Instead, the value of y remains the same (or, in statistics, approximately the same) no matter what x is.

The method described here was chosen for several reasons. One purpose of exploring the first and second order differences is to lay the underpinnings of the first and second derivatives many students will meet in calculus. Another purpose is for students to discover that the differences of second-degree functions are first degree, and that the differences of first-degree functions are constant (degree zero). Still a third purpose is for students to make some initial attempts at connecting mathematics and physics.

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“BUSTING BARRIERS” WITH THE ALGEBRA FX 2.0

The “Algebra” menu on the ALGEBRA FX2.0 can be used to solve quadratic equations or to factor quadratic expressions. For example, suppose we wanted to determine at what time our object would be 50 centimeters from the bottom. To do so, we need to solve the equation $-84.89x^2 + 125.61x + 38.92 = 50$. From the MAIN MENU on the ALGEBRA FX2.0:

- x Call up the “Algebra” menu.
- x Press **F1** for TRNS.
- x Press the appropriate key for SOLVE.
- x Type the equation in, a comma, and then an X. Close the parentheses.
- x Press **EXE** to see the two exact solutions. (Use the right arrow to scroll to the end.)
- x For a decimal approximation of these, press **F1** for TRNS, the appropriate key for APPROXIMATE, show the expression, and press **EXE** .

Using the TUTORIAL Menu

The ALGEBRA FX2.0 can also help students learn how to solve quadratic equations. From the MAIN MENU:

- x Call up the “Tutorial” menu.
- x Highlight “Quadratic Equation” and press **EXE** .
- x Highlight the form that you want, and press **F2** to input the constants.
- x Type in values for the constants, pressing **EXE** after each. Then press **F6** for “Execute.”
- x Press **F6** for the calculator to steer you through the solution automatically.
- x Use **F6** to move to the next step or **F1** to back up a step.