CLEMSON ALGEBRA PROJECT
UNIT 15: COMPLEX NUMBERS

PROBLEM 1: PRISONERS AND ESCAPEES

The study of fractals is relatively new to the world of mathematics. Fractals are figures that are self-similar; that is, part of the figure cannot be distinguished from the whole figure. For those interested in art and the beauty of mathematics, two exciting areas to explore are the Mandelbrot set and Julia sets, sets named after the mathematicians credited with first exploring them. Many Internet Web sites explore these sets and show some of the fascinating art generated by them.

A key mathematical idea for fractals is that of iteration. To perform an iteration, begin with some input value and evaluate it for some specified activity or function. The output then becomes the input for the next iteration, and the process is continued. For example, consider the function $f(x) = x^2$ with an initial input value, called a seed, of 2. First, $f(2) = 4$. Then, $f(4) = 16$, $f(16) = 256$, $f(256) = 65536$, and so on. Note how the outputs are growing without bound; they diverge. Also note that if we begin with an input between 0 and 1, say 0.5, the outputs fairly quickly approach 0.

The Julia set looks at iterations of such a function for different seed values. Three results are possible: the outputs may converge (prisoners!), the outputs may diverge (escapees!), or the outputs may bounce around chaotically. The Julia set consists of the points that bounce around chaotically. The art associated with the Julia set is based on coloring the results of different seeds. Seeds that ultimately diverge are colored green; seeds that ultimately converge are colored red; and seeds that do neither are colored black. Shading is based on the number of iterations it takes for the seed to “decide.”

For this exploration, consider the unit circle drawn in the complex plane and the function $f(Z) = Z^2$. Choose values inside, outside, and on the unit circle. Make a conjecture as to which points in the complex plane are prisoners (converge), which are escapees (diverge), and which do neither (members of the Julia set).

MATERIALS

Casio CFX-9850Ga PLUS or ALGEBRA FX2.0 Graphing Calculator

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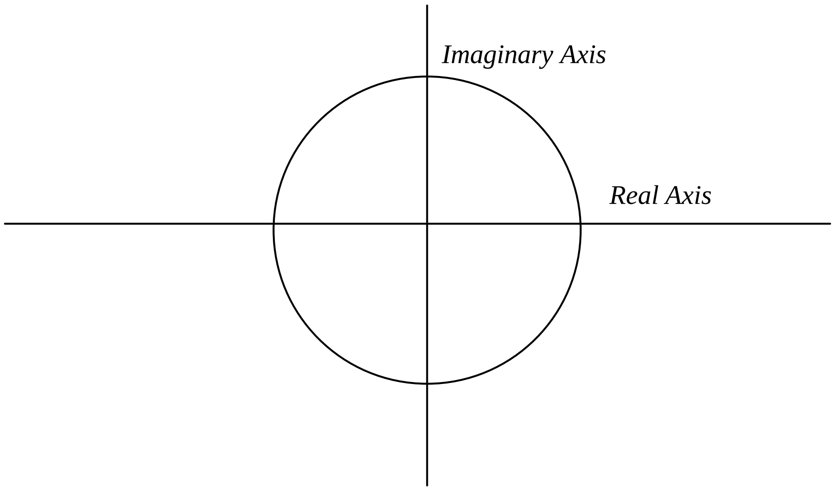
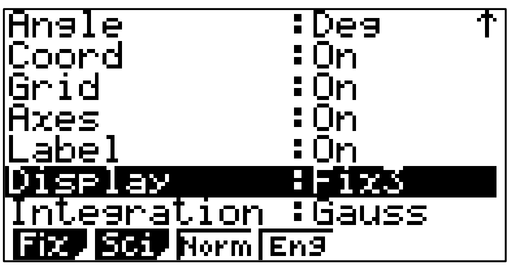
ONE SOLUTION TO PROBLEM 1: PRISONERS AND ESCAPEES



To begin, first consider the unit circle in the complex plane. We wish to look at points inside, outside, and on the circle. This means we need to find points whose distance from the origin (their absolute value) is less than, greater than, or equal to one. With your calculator,

- From the MAIN MENU, choose "Run."
- So we will be able to see the display, we'll fix the decimal places at 3. Press [SHIFT] [MENU] for SET UP. Use the arrow keys to move down to Display, press [F1] for Fix, and [F4] for 3. See below left. Press [EXIT] .
- Press [OPTN] and [F3] for "COMPLEX." This will allow us to easily type in **i** or find other characteristics and perform operations on complex numbers.

Let's begin with a number inside the circle, say **.6 + .7i**. To verify that this number is indeed inside the circle, check its absolute value by using the Abs ([F2]) function on the screen. See below right to confirm that this distance is indeed less than 1.



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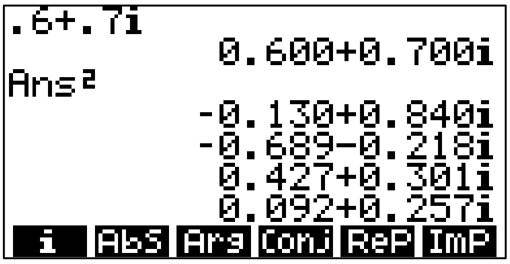
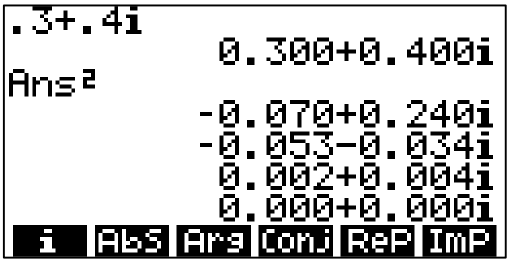
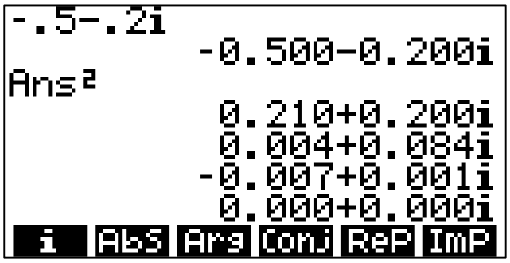
We now will use this input to iterate the function $f(Z) = Z^2$. To do so,

- Press AC/ON to clear the screen.
- Type in the number using F1 for **i**.
- Press EXE to "seed" the calculator with this number.
- Press the squaring key and EXE to square the number that has been seeded.
- Continue pressing EXE to obtain further iterations. See the screen below.
- Note that we quickly get to **0 + 0i** and eventually just 0. This point is a prisoner!



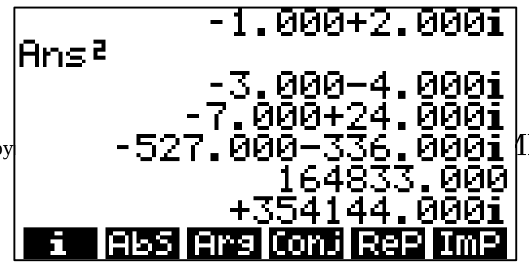
Students should pick other complex numbers within the unit circle. Initial iterations from two additional points, **.3 + .4i** and **-.5 - .2i**, are shown below.

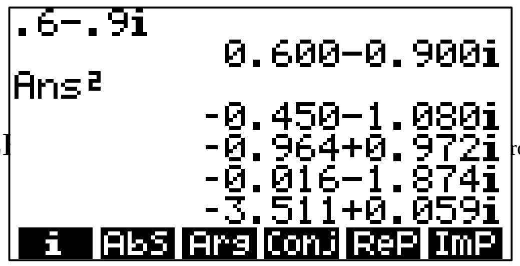




Students may readily conclude that points within the unit circle are prisoners; in other words, they cannot escape, converging fairly quickly to 0. For those who are interested in the art associated with the Julia sets, these points would be colored red.

Similar exploration should be conducted with points outside the unit circle. The screens below show the results from iterations with initial seeds of **–1 + 2i** and **.6 - .9i**.





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COMPLEX NUMBERS

Students should recognize that the first point is an escapee; it diverges rather quickly. Note that the calculator takes two lines to display the last complex number shown.

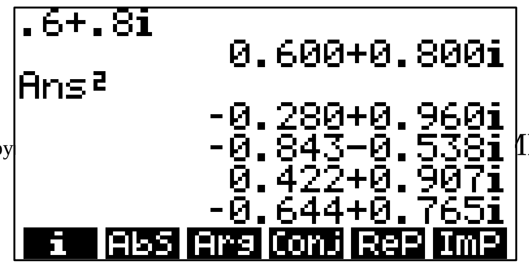
Whether **.6 - .9i** converges or diverges is not readily apparent from the display shown. However a few more iterations should convince students that this number does indeed escape from the unit circle also; it merely takes a little longer to do so.

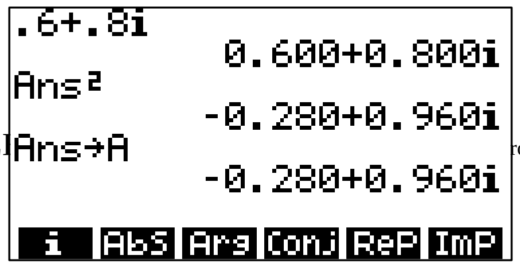
Based on these and their findings from other points, students may make the conjecture that numbers outside the unit circle are all escapees. They may further surmise that points farther away from the circle diverge more quickly than those closer to the circle. In other words, iterations on the square function diverge for seeds with absolute value greater than 1.

What is left to explore is what happens to points that begin on the unit circle. This area is more problematic due to the necessary rounding that the calculator must do. Suppose students start by iterating the point **.6 + .8i**. (Check its absolute value to verify that it is on the unit circle.) Shown below left are the first few iterations of .6 + .8i, which do seem to bounce chaotically. Continued iterations, though, will suggest convergence, as the values settle to 0. However, if we could perform the squaring operations without rounding, we would find that the points actually continue to bounce around forever, making these points part of the Julia set. This is an important lesson for students; calculators are exciting and powerful devices, but they are not perfect.

To help them see that these points actually do bounce around on the unit circle instead of converging to 0, students should try plotting the first several iterations in the complex plane. With assistance, they may recognize that the iterations do indeed bounce around, but all of the points are located on the unit circle itself. Unlike the points inside the unit circle, they do not move to 0.

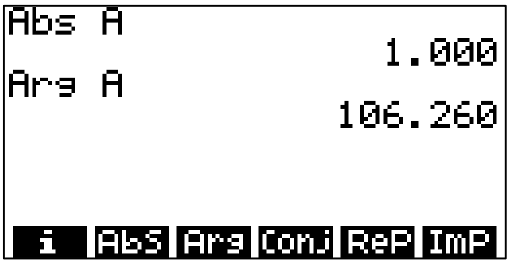
One way to help students recognize this is to take the absolute value of the various iterations and to determine the angle at which the point should be plotted. First, though, it will be helpful to store the first few iterations. After seeding **.6 + .8i**, square the result and store this result in memory location A. (The STORE key is the right arrow key

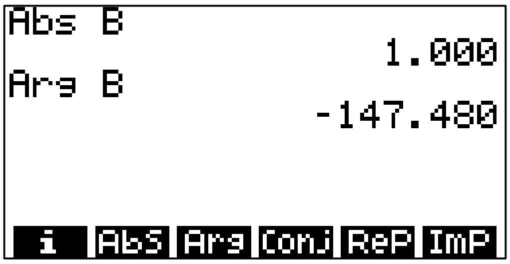




above AC/ON . Square A and store the result in memory location B. The first steps are shown below right.

We are now ready to find the absolute value (distance from 0) and the argument (the angle at which this point is located from standard position) of these iterations. Before doing this, press SHIFT MENU to ensure that, in the SET UP, angles are measured in degrees. Now, simply find the absolute value and argument for points A, B, and as many other iterations as you have done. The screens below shows the results for A and B, respectively.





Interpreting these results may help prepare students for the study of polar coordinates. They indicate that point A lies 1 unit from the origin, and thus is on the unit circle, at an angle of a little more than 106° from standard position. Point B also lies on the unit circle, but at an angle of about -147° from standard position. Continued exploration may help students recognize when the problems with rounding cause the iteration to move off the unit circle.

As a connection to the mathematics with which they are more familiar, the real numbers, you may ask students to discuss what happens when numbers larger than 1 are squared, when numbers less than 1 are squared, and when 1 itself is squared. The fascinating new aspect of this is that as we square complex numbers whose absolute value is 1, we retain the value of 1, but the position of the point bounces chaotically about the unit circle.

Based on these findings, students may indeed conjecture that points on the circle are the members of the Julia set. For the most part, this conjecture is true. However, there are exceptions that students may discover. For example, students may try the point **1 + 0i**,

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which is clearly on the unit circle. They will note that iterations of this point will surely converge at 1. Similarly, they may also note that iterations of **0 + 1i** also converge at 1. In fact, rotations of **1 + 0i** by angles of $360/2^n$, for positive integral values of n, all converge to 1. For example, students may discover that iterations of the point on the unit circle at 45 degrees, which has the value of $\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$, also converge to 1. This aspect of the investigation, however, is most likely beyond the scope of Algebra I and Algebra II.

Teachers may also wish to tie the idea of squaring these complex numbers to DeMoivre's Theorem, which states that $z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$. In other words, to square a complex number, square the modulus (the absolute value) and double the angle. Exploration of this idea may be an excellent preparation for calculus. (NOTE: When we find the argument of the complex numbers, it is not readily apparent that the angles are doubled each time. This is because the calculator returns a value between positive and negative 180 degrees.)

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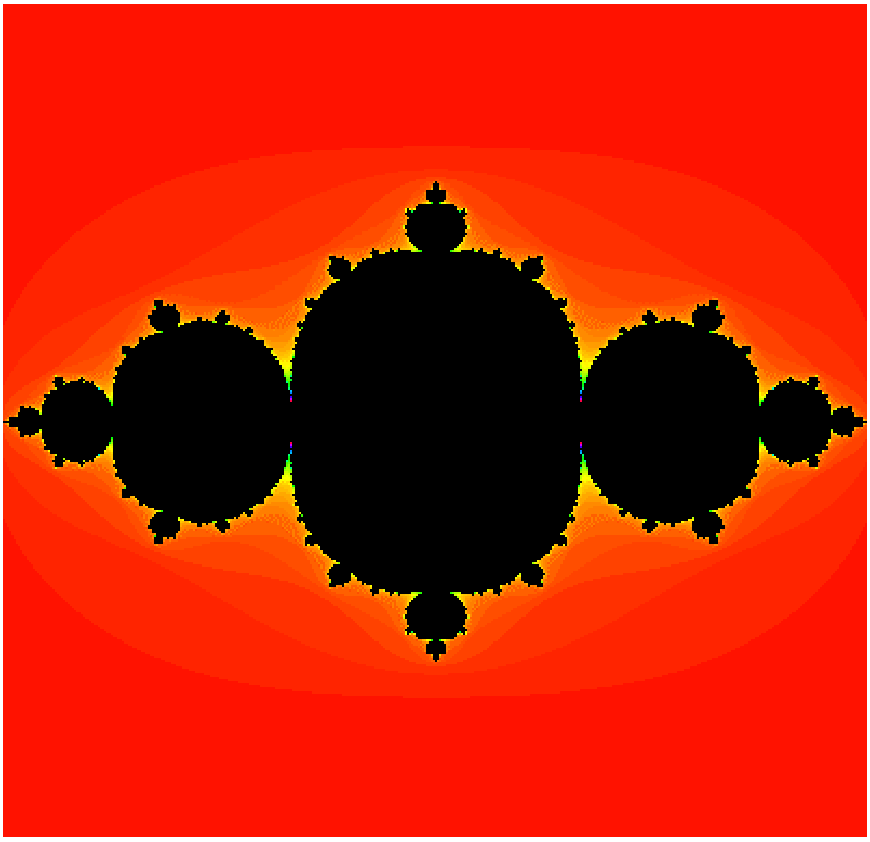
The picture on this page is the Julia set created by the function $f(Z) = Z^2 - .75$. The border of the darkest part is the actual Julia set, and the darkest region is the area enclosed by the Julia set.



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