

Differential Equation Calculation (DIFF EQ) Examples

- 1. Slope Field**
- 2. Calculus - Circle and Cycloid**
- 3. Number Theory - Prime Number Theorem**
- 4. Calculus/Financial - Continuous Compounding**
- 5. Calculus - Exponential Growth and Decay**
- 6. Physics - Pendulum**
- 7. Physics - Spring-mass System**
- 8. Physics - Circuit**
- 9. Physics - Van del Pol Equation**
- 10. Biology - Predator-prey Equations**
- 11. Chaos - Strange Attractor**
- 12. Chemistry - The 'Brusselator'**
 - Numerical Methods of Differential Equation (RECUR Mode)**

1. Slope Field

Any 1st order differential equation $y' = F(x,y)$ admits a geometric interpretation $y' = F(x,y)$ prescribes a tangent line at each point of the plane: the line through (x,y) whose slope is $F(x,y)$. Conversely, the function $F(x,y)$ assigns to each point (x,y) a slope field (direction field or tangent field) of the solution to the differential equation. We can visualize the situation by drawing short line segments whose slope is $F(x,y)$ at various points (x,y) of the plane. This field can be used to approximate graphically the solutions of the differential equations.



Example 1 For each of the following differential equations, Sketch a slope field by hand.
 (1) Confirm your sketch using the DIFF EQ Mode on your Algebra FX calculator.
 (2) What information (if any) can you determine about the solution y as $x \rightarrow \infty$?
 (a) $dy/dx = 1 - y$ (b) $dy/dx = x$ (c) $dy/dx = -y/x$

Answer

- (1) Consider and create your own sketches by hand. See Result Screen.
 (2) (a) y converges on 1. $(y = 1 - \exp(-x + C) ; C \text{ is the integration constant.})$
 (b) y diverges. $(y = x^2 + C)$
 (c) y converges on 0. $(y = C/x)$

Procedure

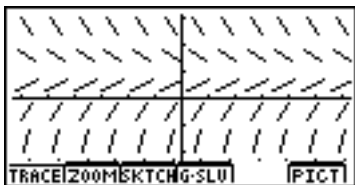
- ① **MENU** DIFF EQ
- ② **F1** (1st) **4** (Others)
- ③ **F4** (DEL·A) **EXE**

In the case of (a):
1 **□** ***1** **ALPHA** **□** **(Y)** **EXE** ($dy/dx = 1 - y$)

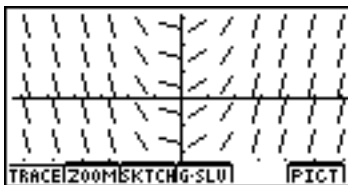
In the case of (b):
X,θ,T **EXE** ($dy/dx = x$)

- In the case of (c):
□ ***2** **ALPHA** **□** **(Y)** **□** **X,θ,T** **EXE**
 ($dy/dx = -y/x$)
 ④ **SHIFT** **OPTN** (V-Window) **F1** (INIT)
 (Initializes V-Window settings.) **ESC**
 ⑤ **F6** (CALC)

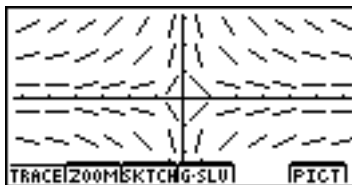
Result Screen



(a) $dy/dx = 1 - y$



(b) $dy/dx = x$



(c) $dy/dx = -y/x$



*1*2 Do not confuse the **□** key and the **□** key. as the subtraction symbol.
 An error occurs if you use the **□** key



Example 2 (1) Choose a solution curve of the initial-value problem $y' = 2 - y$, $y(0) = 1$ from the graphs in Figure 1.
 (2) Confirm your choice using the DIFF EQ Mode on your Algebra FX calculator.

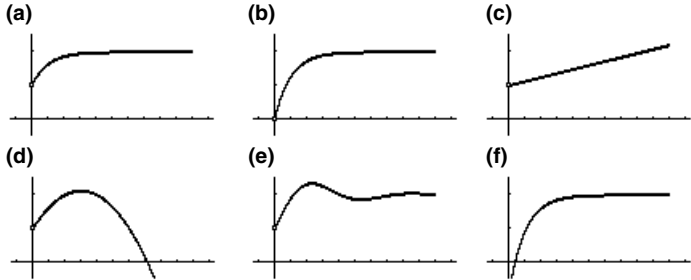


Figure 1 Possible solution curves

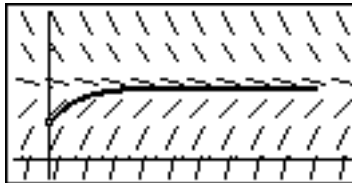
Answer

- (1) (a)
- (2) See Procedure.

Procedure

- | | |
|---|--|
| <ul style="list-style-type: none"> ① MENU DIFF EQ ② F1 (1st) 4 (Others) ③ 2 ← ALPHA ← (Y) EXE ($dy/dx = 2 - y$) ④ 0 EXE ($x_0 = 0$) 1 EXE ($y_0 = 1$) ⑤ F5 (SET) 1 (Param) F1 (INIT)
 0 EXE 1 0 EXE ($0 \leq x \leq 10$)
 0 • 0 5 EXE ($h = 0.05$) ESC | <ul style="list-style-type: none"> ⑥ F5 (SET) 2 (Output) F4 (INIT) ESC ⑦ SHIFT OPTN (V-Window) F1 (INIT)
 ← 1 • 3 EXE ($Xmin = -1.3$)
 1 1 • 3 EXE ($Xmax = 11.3$) ▼ ▼
 ← 0 • 5 EXE ($Ymin = -0.5$)
 4 EXE ($Ymax = 4$) ESC ⑧ F6 (CALC) CTRL 0 *1 |
|---|--|

Result Screen



*1 Press **CTRL** **0** to toggle display of the menu at the bottom of the screen on and off.

2. Calculus - Circle and Cycloid

■ The Circle

Let (a, b) be the center of the circle, and r be its radius. The equation of the circle is

$$(x - a)^2 + (y - b)^2 = r^2. \quad (1)$$

Implicit differentiation of the equation with respect to x yields

$$2(x - a) + 2(y - b) \cdot dy/dx = 0. \quad (2)$$

Solve for dy/dx , and we obtain the differential equation of the circle

$$dy/dx = -(x - a)/(y - b). \quad (3)$$

The parametric equations for the circle are

$$\begin{aligned} x &= r \sin \theta + a, \\ y &= r \cos \theta + b. \end{aligned} \quad (4)$$

Differentiate (4) with respect to θ .

$$\begin{aligned} dx/d\theta &= r \cdot \cos \theta, \\ dy/d\theta &= -r \cdot \sin \theta. \end{aligned} \quad (5)$$

Eliminate $a \cdot \sin \theta$ and $a \cdot \cos \theta$ from the equations (4) and (5), and we obtain the system of the differential equations of the circle

$$\begin{aligned} dx/d\theta &= y - b, \\ dy/d\theta &= -(x - a). \end{aligned} \quad (6)$$

■ The Cycloid

A cycloid is the curve traced by a point on the circumference of a circle rolling along a straight line without slipping. Let r be the radius of the circle.

The parametric equations for the cycloid are

$$\begin{aligned} x &= r (\theta - \sin \theta), \\ y &= r (1 - \cos \theta). \end{aligned} \quad (7)$$

Differentiate (7) with respect to θ .

$$\begin{aligned} dx/d\theta &= r (1 - \cos \theta), \\ dy/d\theta &= r \sin \theta. \end{aligned} \quad (8)$$

Eliminate $a \cdot \sin \theta$ from the equations (7) and (8), and we obtain the system of differential equations of the cycloid,

$$\begin{aligned} dx/d\theta &= y, \\ dy/d\theta &= r \cdot \theta - x. \end{aligned} \quad (9)$$

Circle



- Example** (1) Suppose that trajectory $(x(\theta), y(\theta))$ of a particle moving in the plane satisfies the initial value problem
 $dx/d\theta = y, \quad dy/d\theta = -x, \quad \theta_0 = 0, \quad x(\theta_0) = 2, \quad y(\theta_0) = 0.$
 (2) Graph the trajectory.

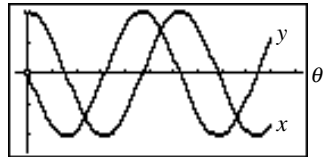
Answer

- (1) $x^2 + y^2 = 2^2.$
 (2) See Result Screen.

Procedure

- ① **MENU** DIFF EQ
- ② **F4** (SYS) **F2** (2)
- ③ **F3** (y_n) **2** **EXE** ($y_1' = y_2$)
← **F3** (y_n) **1** **EXE** ($y_2' = -y_1$)
0 **EXE** ($x_0 = 0$) **2** **EXE** ($(y_1)_0 = 2$)
0 **EXE** ($(y_2)_0 = 0$)
- ④ **F5** (SET) **1** (Param) **F1** (INIT)
0 **EXE** **1** **0** **EXE** ($0 \leq x \leq 10$)
0 **▸** **1** **EXE** ($h = 0.1$) **ESC**
- ⑤ **F5** (SET) **2** (Output) **F4** (INIT) **▼** **▼**
F1 (SEL)(Select (y_1) and (y_2) to graph)
▲ **F2** (LIST) **1** **EXE** (Select LIST1 to store the values for (y_1) in LIST1)
▼ **F2** (LIST) **2** **EXE** (Select LIST2 to store the values for (y_2) in LIST2) **ESC**

- ⑥ **SHIFT** **OPTN** (V-Window) **F1** (INIT)
← **1** **▸** **3** **EXE** (Xmin = - 1.3)
1 **1** **▸** **3** **EXE** (Xmax = 11.3) **ESC** *1
- ⑦ **F6** (CALC)
F2 (ZOOM) **5** (Auto)*2 **CTRL** **0** *3



*1 Here, Ymin and Ymax are not specified. Adjust Ymin and Ymax after drawing the graph.
 *2 In the DIFF EQ Mode, perform **F2** (ZOOM) **5** (Auto) to adjust the V-Window y-axis so the entire solution

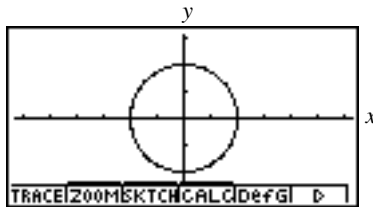
curve fits in the screen along the y-axis.
 *3 Press **CTRL** **0** to toggle display of the menu at the bottom of the screen on and off.

Now you can graph the relationship between variables x and y based on the results we calculated here.

Procedure

- ① **MENU** STAT (List 1 and List 2 contain values for (y_1) and (y_2) respectively.)
- ② **F1** (GRPH) **5** (Set)
- ③ **F1** (GPH1) **F2** (xy)
 - ▼ **F1** (LIST) **1** **EXE**
(XList = List1: (y_1))
 - ▼ **F1** (LIST) **2** **EXE**
(YList = List2: (y_2))
 - ▼ ▼ **F3** (■) **ESC**
- ④ **CTRL** **F3** (SET UP) **F2** (Man)^{*1} **ESC**
- ⑤ **SHIFT** **OPTN** (V-Window) **F1** (INIT) **ESC**
- ⑥ **F1** (GRPH) **1** (S-Gph1)

Result Screen



This is a circle.

$$(x^2 + y^2 = 2^2.)$$



^{*1}In the STAT Mode, use **CTRL** **3** (SET UP) to display the SET UP variable, and then use **F2** (Man) to change the StatWind setting to Manual. After you do this, the V-Window

shows the values used for settings made using **SHIFT** **OPTN** (V-Window). You cannot adjust Xmin, Xmax, Ymin, and Ymax on this screen.

■ Cycloid



Example Suppose that trajectory $(x(\theta), y(\theta))$ of a particle moving in the plane satisfies the initial value problem

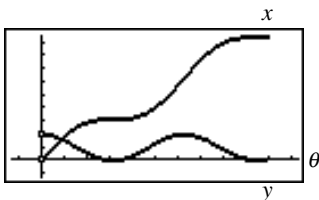
$$dx/d\theta = y, \quad dy/d\theta = \theta - x, \quad \theta_0 = 0, \quad x(\theta_0) = 0, \quad y(\theta_0) = 2.$$

Graph the trajectory $(x(\theta), y(\theta))$.

Procedure

- ① **MENU** DIFF EQ
- ② **F4** (SYS) **F2** (2)
- ③ **F3** (y_n) **2** **EXE** ($y_1' = y_2$)
 x, θ_1 **=** **F3** (y_n) **1** **EXE** ($y_2' = x - y_1$)
- ④ **0** **EXE** ($x_0 = 0$) **0** **EXE** ($(y_1)_0 = 0$)
2 **EXE** ($(y_2)_0 = 2$)
- ⑤ **F5** (SET) **1** (Param) **F1** (INIT)
0 **EXE** **1** **0** **EXE** ($0 \leq x \leq 10$)
0 **.** **1** **EXE** ($h = 0.1$) **ESC**
- ⑥ **F5** (SET) **2** (Output) **F4** (INIT) **▼** **▼**
F1 (SEL)(Select (y_1) and (y_2) to graph.)
▲ **F2** (LIST) **1** **EXE** (Select LIST1 to store the values for (y_1) in LIST1.)
▼ **F2** (LIST) **2** **EXE** (Select LIST2 to store the values for (y_2) in LIST2.) **ESC**

- ⑦ **SHIFT** **OPTN** (V-Window) **F1** (INIT)
 \leftarrow **1** **.** **3** **EXE** ($X_{\min} = -1.3$)
1 **1** **.** **3** **EXE** ($X_{\max} = 11.3$) **ESC** *1
- ⑧ **F6** (CALC)
F2 (ZOOM) **5** (Auto)*2 **CTRL** **0** *3



*1 Here, Ymin and Ymax are not specified. Adjust Ymin and Ymax after drawing the graph.

*2 In the DIFF EQ Mode, perform **F2** (ZOOM) **5** (Auto) to adjust the V-Window y-axis so the entire solution curve fits in the screen along the y-axis.

*3 Press **CTRL** **0** to toggle display of the menu at the bottom of the screen on and off.

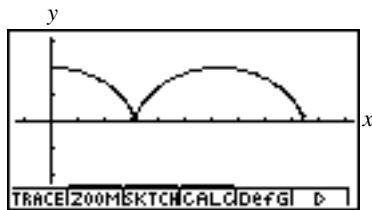
Turning off the function menu display makes it possible to view part of a graph hidden behind it.

Now you can graph the relationship between variables x and y based on the results we calculated here.

Procedure

- | | |
|--|---|
| <p>① MENU STAT (List 1 and List 2 contain values for (y_1) and (y_2) respectively.)</p> <p>② F1 (GRPH) S (Set)</p> <p>③ F1 (GPH1) ▼ F2 (xy)</p> <p style="padding-left: 20px;">▼ F1 (LIST) 1 EXE (XList = List1: (y_1))</p> <p style="padding-left: 20px;">▼ F1 (LIST) 2 EXE (YList = List2: (y_2))</p> <p style="padding-left: 20px;">▼ ▼ F3 (■) ESC</p> | <p>④ CTRL F3 (SET UP) F2 (Man)*¹ ESC</p> <p>⑤ SHIFT OPTN (V-Window) F1 (INIT)</p> <p style="padding-left: 20px;">← 1 ▢ 3 EXE (Xmin = - 1.3)</p> <p style="padding-left: 20px;">1 1 ▢ 3 EXE (Xmax = 11.3)</p> <p style="padding-left: 20px;">ESC (Ymin = - 3.1, Ymax = 3.1)</p> <p>⑥ F1 (GRPH) 1 (S-Gph1)</p> |
|--|---|

Result Screen



This is the cycloid.



*¹In the STAT Mode, use **CTRL** **3** (SET UP) to display the SET UP variable, and then use **F2** (Man) to change the StatWind setting to Manual. After you do this, the V-Window

shows the values used for settings made using **SHIFT** **OPTN** (V-Window). You cannot adjust Xmin, Xmax, Ymin, and Ymax on this screen.

3. Number Theory - Prime Number Theorem

In his study of prime numbers, Gauss came up with a conjecture that is equivalent to the prime number theorem. For a positive real number x , let $\pi(x)$ be the number of primes not exceeding x . The Prime Number Theorem says:

$$\pi(x)/(x/\ln x) \rightarrow 1 \quad (x \rightarrow \infty).$$

It can be said that the average distance between the two consecutive prime numbers on either side of x is $\ln x$.

Let's start with a general explanation of the Prime Number Theorem. Of all the integers not exceeding x , about half are odd numbers that cannot be evenly divided by 2. About 2/3 of these odd numbers cannot be divided evenly by 3. Of the remaining values, about 4/5 cannot be divided evenly by 5. All of this means that the number of primes not exceeding $\Pi(x)$ can be determined by dividing by all prime numbers p , which makes $\pi(x)$ approximately equivalent to $\Pi(1 - 1/p) \cdot x$. Actually, $\Pi(1 - 1/p)$ is approximately equivalent to $1/\ln x$. (Though this is difficult to prove.) Assuming that this is true, we can say that $\pi(x)$ is approximately equivalent to $x/\ln x$. This leads us to the prime number theorem.

The prime number theorem is best method of approximation. Using the prime number density function $\pi(x)/x$

$$\lim_{x \rightarrow \infty} \pi(x)/x = 1/\ln x.$$

This formula implies that average prime number density is equal to $1/\ln x$. This $1/\ln x$ is integrated in the range $[1, x]$.

(Remember that the smallest prime number is 2.)

$$\pi(x) \doteq \text{Li}(x) = \int_2^x dt/\ln t.$$

$\text{Li}(x)$ is called the logarithmic integral function, which uses logarithmic integration for even greater approximation of $\pi(x)$ than $x/\ln x$.

This formula can be rewritten to the following differential equation,

$$y'(x) = 1/\ln(x), \quad y(2) = 0.$$

The distribution of primes $\pi(x)$

x	$\pi(x)$	$x/\ln(x)$	$\text{Li}(x)$	$x/(\ln(x)-1.08366)^*$
10	4	4.3	6.2	8.2
100	25	21.7	30.1	28.4
1000	168	144.8	177.6	171.7
10000	1229	1085.7	1246.1	1230.5
100000	9592	8685.9	9629.8	9588.4
1000000	78498	72382.4	78627.5	78543.2
10000000	664579	620420.7	664918.4	665139.7
100000000	5761455	5428681.0	5762209.4	5768003.7

*According to Legendre

•••••
Example

The 800th prime number is 6,133. This means there are 799 prime numbers less than 6,133. Using this information, determine approximately how many prime numbers there are up to 6,233. Use the following differential equation initial value problem:

$$y' = 1/\ln(x), \quad y(6133) = 799.$$

Use a list of prime numbers to perform your own calculation to check your calculated value against the actual number of prime numbers up to 6,233.

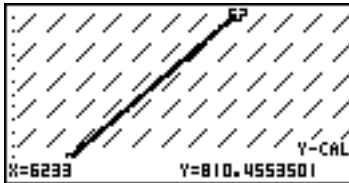
Solution

Approximately 811

Procedure

- | | |
|--|--|
| <p>① MENU DIFF EQ</p> <p>② F1(1ST) 4 (Others)</p> <p>③ 1 ÷ ln X,θT EXE ($y' = 1/\ln(x)$)</p> <p>④ 6 1 3 3 EXE ($x_0 = 6133$)</p> <p style="padding-left: 20px;">7 9 9 EXE ($y_0 = 799$)</p> <p>⑤ F5(SET) 1 (Param) F1(INIT)</p> <p style="padding-left: 20px;">6 1 3 3 EXE 6 2 3 3 EXE</p> <p style="padding-left: 20px;">($6133 \leq x \leq 6233$)</p> <p style="padding-left: 20px;">0 ▸ 5 EXE ($h = 0.5$) ESC</p> <p>⑥ F5(SET) 2 (Output) F4(INIT)</p> <p style="padding-left: 20px;">(Select y to graph.)</p> | <p>F2(LIST) 1 EXE (Select LIST1 to store the values for x in LIST1.) ESC</p> <p>⑦ SHIFT OPTN (V-Window) F1(INIT)</p> <p style="padding-left: 20px;">6 1 0 0 EXE 6 3 0 0 EXE</p> <p style="padding-left: 20px;">1 EXE ESC</p> <p>⑧ F6(CALC) F2(ZOOM) 5 (Auto)*1</p> <p>⑨ F4(G-SLV) 6 (Y-Cal)*2</p> <p style="padding-left: 20px;">6 2 3 3 EXE</p> <p style="padding-left: 20px;">(Determines y when $x = 6233$.)</p> |
|--|--|

Result Screen



*1 After graphing the differential equation, perform **F2**(ZOOM) **5**(Auto) to adjust the V-window y-axis so that entire solution curve fits in the screen along the y-axis.

*2 **F4**(G-SLV) **6** (Y-Cal) calculates the value of y for any value of x on the graphed solution curve $f(x) = y$.

Also:

F4(G-SLV) **1** (Root): Solves for $f(x) = 0$ from a graphed solution curve.

F4(G-SLV) **2** (Max): Calculates the local maximum of a graphed solution curve.

F4(G-SLV) **3** (Min): Calculates the local minimum of a graphed solution curve.

F4(G-SLV) **4** (Y-Intcpt): Calculates the Y-intercept of a graphed solution curve.

F4(G-SLV) **5** (Isect): Determines the intersects when two or more solution curves are graphed.

F4(G-SLV) **7** (X-Cal): Calculates the value of x for any value of y on the graphed solution curve $f(x) = y$.

4. Calculus/Financial - Continuous Compounding

■ Explanation Using a Differential Equation

Let y be the value at time x of y_0 invested at the interest rate r .

Interest is compounded continuously. It means that the instantaneous rate of change of y with respect to time x is ry , that is

$$dy/dx = ry, \quad y(0) = y_0.$$

This is the initial-value problem of 1st order differential equation. Its solution is

$$y = y_0 e^{rx}.$$

Thus, continuous compounding at rate r for a time period x results in growth by a factor of e^{rx} .

■ Explanation Using a Differentiation

If we put y_0 dollars in the bank at $100r\%$ compounded n times per year, it will be $y(x)$ dollars at the end at x years, where

$$y(x) = y_0(1 + r/n)^{nx}.$$

Consider what happens when interest is compounded continuously, that is, the limiting case as the number of compounding periods n goes to infinity. This gives us,

$$y(x) = \lim_{n \rightarrow \infty} y_0(1 + r/n)^{nx} = \lim_{n \rightarrow \infty} y_0[(1 + r/n)^{n/r}]^{rx} = \lim_{h \rightarrow \infty} y_0[(1 + 1/h)^h]^{rx} = y_0 e^{rx}.$$

This is the solution of the initial value problem

$$dy/dx = ry, \quad y(x_0) = y_0.$$

■ Explanation Using Inference

Suppose that you invest 1 dollar for 1 year at 100% annual interest rate.

What is your balance after 1 year and x years?

The answer depends on the compounding schedule the bank uses, as shown by the table below.

Compounding Schedule	n	$y_0(1 + r/n)^n$ ($r = 100\% = 1$)	Balance after 1 year	Balance after year x
Annually	1	$1(1+1/1)^1$	2.00	2.00^x
Semi-annually	2	$1(1+1/2)^2$	2.25	2.25^x
Quarterly	4	$1(1+1/4)^4$	2.4414...	$2.4414...^x$
Monthly	12	$1(1+1/12)^{12}$	2.6130...	$2.6130...^x$
Daily	365	$1(1+1/365)^{365}$	2.7145...	$2.7145...^x$
Hourly	8760	$1(1+1/8760)^{8760}$	2.7182812...	$2.7182812...^x$
Every second	31536000	$1(1+1/31536000)^{31536000}$	2.7182816...	$2.7182816...^x$
...
Continuously	∞	$\lim_{n \rightarrow \infty} 1(1+1/n)^n$	$e=2.7182818...$	e^x

As the number of compounding periods n increases, the balance approaches a limit of e . Thus, continuous compounding at rate r for time period x results in growth by a factor of e^{rx} .



Example Consider a savings account that initially contains \$100, and earns interest at the annual rate 3% compounded continuously. Assume that deposits are added at \$10 per year continuously.¹ The number of dollars y in the account at time x (years) will satisfy the initial value problem

$$dy/dx = 10 + 0.04y, y(0) = 100.$$

About how many years will it take for the amount in the account to reach \$250?

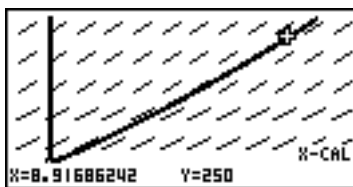
Answer

Approximately 8.9 years

Procedure

- | | |
|--|--|
| <p>① MENU DIFF EQ</p> <p>② F1 (1st) 4 (Others)</p> <p>③ 1 0 + 0 . 0 4
 ALPHA (Y) EXE ($y' = 10 + 0.04y$)</p> <p>④ 0 EXE ($x_0 = 0$)
 1 0 0 EXE ($y_0 = 100$)</p> <p>⑤ F5 (SET) 1 (Param) F1 (INIT)
 0 EXE 1 0 EXE ($0 \leq x \leq 10$)
 0 . 0 5 EXE ($h = 0.05$) ESC</p> | <p>⑥ F5 (SET) 2 (Output) F4 (INIT)
 (Select y to graph.) ESC</p> <p>⑦ SHIFT OPTN (V-Window) F1 (INIT)
 (-) 1 . 3 EXE ($Xmin = -1.3$)
 1 1 . 3 EXE ($Ymax = 11.3$) ESC</p> <p>⑧ F6 (CALC) F2 (ZOOM) 5 (Auto)²</p> <p>⑨ F4 (G-SLV) 7 (X-Cal)³
 2 5 0 EXE
 (Solve the equation $y = 250$ for x)</p> |
|--|--|

Result Screen



^{*1} This is not done in real-life banking situations.

^{*2} After graphing the differential equation, perform **F2** (ZOOM) **5** (Auto) to adjust the V-Window y -axis so that the entire solution curve fits in the screen along the y -axis.

^{*3} **F4** (G-SLV) **7** (X-Cal) calculates the value of x for any value of y on the graphed solution curve $f(x) = y$.

Also

F4 (G-SLV) **1** (Root): Solves for $f(x) = 0$ from a graphed solution curve.

F4 (G-SLV) **2** (Max): Calculates the local maximum of a graphed solution curve.

F4 (G-SLV) **3** (Min): Calculates the local minimum of a graphed solution curve.

F4 (G-SLV) **4** (Y-Intpt): Calculates the Y-intercept of a graphed solution curve.

F4 (G-SLV) **5** (Isect): Determines the intersects when two or more solution curves are graphed.

F4 (G-SLV) **6** (Y-Cal): Calculates the value of y for any value of x on the graphed solution curve $f(x) = y$.

5. Calculus - Exponential Growth and Decay

■ Exponential Growth and Decay

In many applications, the rate of change of a variable y is proportional to the value of y .

If y is a function of time t , then we can write

$$dy/dt = ky.$$

The general solution of this differential equation is

$$y = Ce^{kt}.$$

C is called the initial value of y , and k is called the constant of proportionality.

Exponential growth is indicated by $k > 0$, and *exponential decay* by $k < 0$.

■ Simple Population Model

Let y be the size of the population at some time t .

Let the number of individuals added by birth and lost by death be in proportion to y .

The *simple population model* is

$$dy/dt = ky, \quad y(t_0) = y_0.$$

■ Logistic Equation

Let y be the size of the population at some time t .

Suppose the number of individuals added by birth is in proportion to y .

Suppose the number of individuals lost due to competition is proportional to the square of y .

The population model with birth and competition is

$$dy/dt = ky(A - y), \quad y(t_0) = y_0,$$

where A is its supposed maximum allowable limit and k is a constant. It is often called the *Logistic equation*.^{*1}



*1 The Logistic equation is also used in statistics.

■ Exponential Decay - Radioactive Decay



Example Nitrogen-13, one of the isotopes of nitrogen, is radioactive and decays at a rate proportional to the amount present. Its half-life is about 10 minutes; that is, it takes about 10 minutes for given amount of N-13 to decay one-half its original size. Let's say you have three grams of Nitrogen-13. Graph the change in N-13 weight after 20 minutes.

Solution

The change rate of N-13 satisfies the differential equation

$$dy/dt = ky,$$

where k is negative. The half-life of 10 minutes allows us to determine k , since it implies

$$1/2 = 1e^{k(10)}.$$

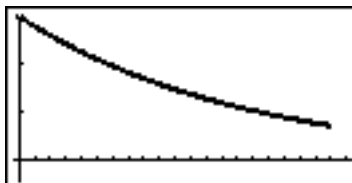
Thus,

$$k = -(\ln 2)/10.$$

Procedure

- ① **MENU** DIFF EQ
- ② **F1** (1st) **4** (Others)
- ③ **(←) (→) ln 2) ÷ 1 0 ×**
ALPHA **=** (Y) **EXE** ($y' = -(\ln 2)/10 \times y$)
- ④ **0** **EXE** ($x_0 = 0$) **3** **EXE** ($y_0 = 3$)
- ⑤ **F5** (SET) **1** (Param) **F1** (INIT)
0 **EXE** **2** **0** **EXE** ($0 \leq x \leq 20$)
0 **•** **1** **EXE** ($h = 0.1$) **▼**
0 **EXE** (No slope field display.)^{*1} **ESC**
- ⑥ **F5** (SET) **2** (Output) **F4** (INIT) **ESC**
- ⑦ **SHIFT** **OPTN** (V-Window) **F1** (INIT)
(←) 1 • 3 **EXE** (Xmin = -1.3)
2 1 • 3 **EXE** (Xmax = 21.3) **▼** **▼**
(→) 0 • 5 **EXE** (Ymin = -0.5)
3 **EXE** (Ymax = 3) **ESC**
- ⑧ **F6** (CALC) **CTRL** **0**^{*2}

Result Screen



This is called *exponential decay*.



*1 If you do not want to display a slope field, make SF equal 0.

*2 Press **CTRL** **0** to toggle display of the menu at the bottom of the screen on and off.

Logistic Equation



Example Graph the slope field and the solution of the differential equation

$$y' = y(2 - y), -5 \leq x \leq 5, x_0 = 0,$$

$$y_0 = \{0.5, 1, 1.5\}^*, h = 0.1.$$

Use the following V-Window settings.

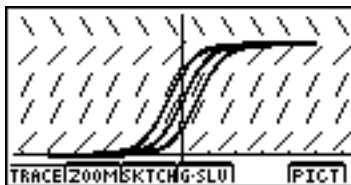
$$\mathbf{Xmin} = -6.3, \mathbf{Xmax} = 6.3, \mathbf{Xscale} = 1,$$

$$\mathbf{Ymin} = -0.5, \mathbf{Ymax} = 2.5, \mathbf{Yscale} = 1.$$

Procedure

- | | |
|--|--|
| <p>① MENU DIFF EQ</p> <p>② F1 (1st) 4 (Others)</p> <p>③ ALPHA = (Y) C 2 = ALPHA = (Y) 7 EXE
 $(y' = y(2 - y))$</p> <p>④ 0 EXE ($x_0 = 0$)
 SHIFT X (f) 0 . 5 . 1 . 1 . 5
 SHIFT + (f) EXE ($y_0 = \{0.5, 1, 1.5\}^*$)</p> <p>⑤ F5 (SET) 1 (Param) F1 (INIT)
 (←) 5 EXE 5 EXE ($-5 \leq x \leq 5$)
 0 . 1 EXE ($h = 0.1$) ESC</p> | <p>⑥ F5 (SET) 2 (Output) F4 (INIT) ESC</p> <p>⑦ SHIFT OPTN (V-Window) F1 (INIT)
 (←) 6 . 3 EXE ($Xmin = -6.3$)
 6 . 3 EXE ($Xmax = 6.3$)
 1 EXE ($Xscale = 1$)
 ▼ (←) 0 . 5 EXE ($Ymin = -0.5$)
 2 . 5 EXE ($Ymax = 2.5$)
 1 EXE ($Yscale = 1$) ESC</p> <p>⑧ F6 (CALC)</p> |
|--|--|

Result Screen



This is called a *logistic curve*.



*1 To graph a family of solutions, enter a list of initial conditions.

6. Physics - Pendulum

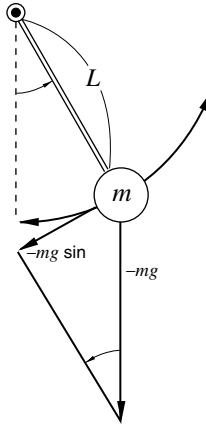


Figure 1

Figure 1 shows a diagram of pendulum. A mass m is suspended from an arm of length L , whose mass we will disregard. Assume the arm is attached to a pivot that is frictionless. The angle of the arm from the vertical is denoted θ , and it is measured counterclockwise in radians. The complete model for the pendulum is

$$L\theta'' + g \sin \theta = 0, \quad \theta(0) = \theta_0, \quad \theta'(0) = \omega_0. \quad (1)$$

Physics allows us to say the following. If we assume that θ is small, we can use the approximation $\sin \theta \approx \theta$, where θ is measured in radians. Therefore, the approximate model for the pendulum becomes

$$L\theta'' + g\theta = 0, \quad \theta(0) = \theta_0, \quad \theta'(0) = \omega_0. \quad (2)$$

The model (2) has the exactly the same mathematical form as the undamped spring-mass model.

Denote that ω is angular velocity, the pendulum model (2) is written as a system of 1st order differential equations

$$\begin{aligned} \theta' &= \omega, \quad \theta(0) = \theta_0, \\ \omega' &= -g/L \sin \theta, \quad \omega(0) = \omega_0. \end{aligned}$$

•••••
Example

A pendulum of length 0.5m and mass 0.1kg is released from rest when the cord makes an angle of π (in radians) with the vertical.

(1) Write a differential equation that relates angle θ , angular velocity ω , and time t .

(2) Graph the solution θ of (1)

$\theta(0) = \pi$ (in radians)*¹, $\omega(0) = 0$ (radians/sec.)*¹,
 $0 \leq t \leq 10$, $h = 0.1$.

Use the following V-Window settings.

Xmin = -1.3, Xmax = 11.3, Xscale = 1,

Ymin = -5.0, Ymax = 5.0, Yscale = 1.

What would the shape of the curve be in Problem (2) if initial condition $\theta(0)$ is 0.999π , 0.8π ?

Answer

(1) $\theta' = \omega$, $\theta(0) = \pi$

$\omega' = -9.8/0.5 \sin \theta$, $\omega(0) = 0$.

(2) See Result Screen.

Procedure

- ① **MENU** DIFF EQ
- ② **F4** (SYS) **F2** (2)
- ③ **F3** (y_1) **2** **EXE** ($y_1' = y_2$)
(←) 9 . 8 ÷ 0 . 5 X
sin F3 (y_1) **1** **EXE**
($y_2' = -9.8/0.5 \times \sin(y_1)$)
- ④ **0** ($x_0=0$) **EXE**
 - (a) **SHIFT EXP** (π) **EXE** (When $(y_1)_0 = \pi$)
 - (b) **0 . 9 9 9** **SHIFT EXP** (π) **EXE**
(When $(y_1)_0 = 0.999\pi$)
 - (c) **0 . 8** **SHIFT EXP** (π) **EXE**
(When $(y_1)_0 = 0.8\pi$)
- 0** **EXE** ($(y_2)_0 = 0$)
- ⑤ **F5** (SET) **1** (Param) **F1** (INIT)
 - 0** **EXE** **1** **0** **EXE** ($0 \leq x \leq 10.0$)
 - 0** **EXE** **1** **EXE** ($h = 0.1$) **ESC**
 - ⑥ **F5** (SET) **2** (Output) **F4** (INIT)
(Select y_1 to graph.) **ESC**
 - ⑦ **SHIFT OPTN** (V-Window) **F1** (INIT)
 - (←) 1 . 3** **EXE** (Xmin = -1.3)
 - 1 1 . 3** **EXE** (Xmax = 11.3)
 - 1** **EXE** (Xscale = 1)
 - ▼ (←) 5** (Ymin = -5.0)
 - 5** **EXE** (Ymax = 5.0)
 - 1** **EXE** (Yscale = 1) **ESC**
 - ⑧ **F6** (CALC)

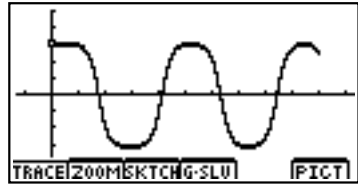


*¹Use RAD (radians) as the angle unit setting.

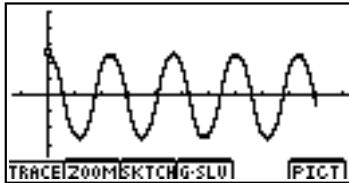
Result Screen



(a) When $\theta(0) = \pi$ (still)



(b) When $\theta(0) = 0.999\pi$
(periodic solution)



(c) When $\theta(0) = 0.8\pi$
(near simple harmonic motion)

Other Things to Do

- Think about why a graph like the one in (a) was produced.
- Try using various initial values for $\theta(0)$ to graph a simultaneous system of equations as solution curves.
- When solving the pendulum differential equation, think about the fact that we used the approximation $\sin \theta \approx \theta$ regardless of the initial conditions.

7. Physics - Spring-mass System

Describing the motion of an oscillating spring is one of the most well-known applications of linear differential equations. It is called the *spring-mass system*.

■ Undamped System

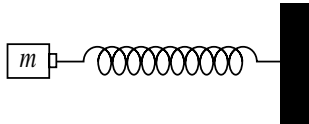


Figure1

Consider a physical system like the one shown in Figure 1, which consists of mass m attached to a spring, where the mass is free to move on a horizontal, frictionless track. Assume that the mass of the spring is negligible compared to m .

Hooke's law tells us that a spring that is stretched (or compressed) y units from its natural length tends to restore itself to its natural length by a force F that is proportional to y . This means,

$$F = -ky,$$

where k is the spring constant or force constant of the spring. It indicates the stiffness of the spring.

According to Newton's Second Law of Motion*¹, the force action on the weight is $F = ma$, where $a = d^2y / dt^2$ is the acceleration. Accordingly,

$$m(d^2y / dt^2) = ma = F = -ky.$$

An initial-value problem of the system can be written as a single 2nd order differential equation,

$$my'' + ky' = 0,$$

$$y(0) = y_0, y'(0) = v_0,$$

or as a system of two 1st order differential equations,

$$y' = v, mv' = -ky,$$

$$y(0) = y_0, v(0) = v_0.$$

These are called *undamped*, because they predict that the mass will oscillate forever once started.

If no external force is applied on the spring-mass system, the system experiences free vibration. System motion is established by initial conditions.



*¹Newton's second law of motion tells us: The effect of an applied force is to cause the body to accelerate in the direction of the force.

Acceleration is, in effect, proportional to the force and inversely proportion to the mass of the body.

■ Damped System

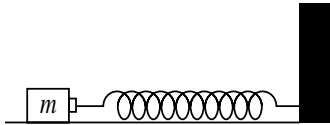


Figure 2

Suppose the object in Figure 2 undergoes additional damping or frictional force proportional to its velocity, such as the damping force due to friction and/or movement through a fluid. Considering this damping force, $-p(dy/dt)$, the differential equation for the oscillations is

$$m(d^2y/dt^2) = -ky - p(dy/dt),$$

or, in standard linear form,

$$(d^2y/dt^2) + (p/m)(dy/dt) + (k/m)y = 0.$$

This means the initial-value problem of the system can be written as a single 2nd order differential equation,

$$y'' + (p/m)y' + (k/m)y = 0,$$

$$y(0) = y_0, y'(0) = v_0,$$

or as a system of two 1st order differential equations,

$$y' = v, \quad v' = -(p/m)v - (k/m)y,$$

$$y(0) = y_0, \quad v(0) = v_0.$$

These are called *damped* because friction can be shown to slow the motion.

•••••
Example

When a mass of 0.32kg is suspended from a spring, the end of the spring moves 2/3m from its natural position. Suppose the weight is pulled 0.5m below the equilibrium position and released.

(1) Write an initial value problem whose solution will describe the position of the mass y at time t .

(2) Graph the solution of the initial value problem.

Set $h = 0.05$, and then determine the movement of y when $0 \leq t \leq 10$.

Use the following V-Window settings.

Xmin = -1.3, Xmax = 11.3,

Ymin = -0.8, Ymax = 0.8.

Answer

(1) $my'' + ky = 0$, $m = 0.32(\text{kg})$, $g = 9.80(\text{N/m})$,

$k = mg/d = 0.32 \times 9.80/(2/3) = 4.704(\text{N/m})$, $y(0) = -0.5(\text{m})$, $v(0) = 0(\text{m/s})$.

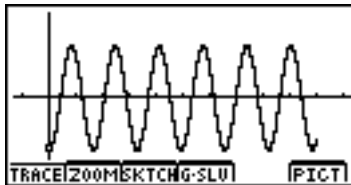
Therefore: $0.32y'' + 4.704y = 0$, $y(0) = -0.5$, $y'(0) = 0$.

(2) See Result Screen.

Procedure

- | | |
|---|--|
| ① MENU DIFF EQ | 0 ▸ 0 5 EXE ($h = 0.05$) ESC |
| ② F2 (2nd) | ⑥ F5 (SET) 2 (Output) F4 (INIT) ESC |
| ③ 0 EXE 4 ▸ 7 0 4 ÷ | ⑦ SHIFT OPTN (V-Window) F1 (INIT) |
| 0 ▸ 3 2 EXE 0 EXE | (←) 1 ▸ 3 EXE (Xmin = -1.3) |
| $(y'' + 4.704/0.32y = 0)$ | 1 1 ▸ 3 EXE (Xmax = 11.3) |
| ④ 0 EXE ($x_0 = 0$) | ▼ ▼ (←) 0 ▸ 8 EXE |
| (←) 0 ▸ 5 ($y(0) = -0.5$) | (Ymin = -0.8) |
| 0 EXE ($y'(0) = 0$) | 0 ▸ 8 EXE (Ymax = 0.8) ESC |
| ⑤ F5 (SET) 1 (Param) F1 (INIT) | ⑧ F6 (CALC) |
| 0 EXE 1 0 EXE ($0 \leq x \leq 10$) | |

Result Screen



Other Things to Do

- Think about why we can apply this differential equation in the case of a mass that is suspended from a spring.

8. Physics - Circuit

■ RL-Circuit

Voltage drop V_R across the resistor in RL -circuit (Figure 1) is RI . Voltage drop V_L across the inductor is $L \cdot di/dt$. According to Kirchhoff's Voltage law (KVL)*1, the sum of these two voltage drops is equivalent to electromotive force $E(t)$.

Thus:

$$L \cdot I' + RI = E(t). \quad (1)$$

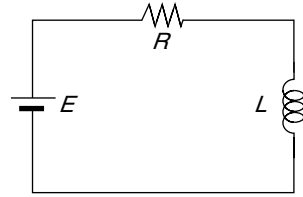


Figure 1

■ RC-Circuit

Voltage drop V_R across the resistor in RC -circuit (Figure 2) is $R \cdot dQ/dt = RI$. Voltage drop V_C across the capacitor is $1/C \cdot Q = 1/C \cdot \int I dt$. According to KVL , the sum of two voltage drops is equivalent to electromotive force $E(t)$.

Thus:

$$RI + 1/C \cdot \int I \cdot dt = E(t), \quad (2)$$

$$R \cdot dQ/dt + 1/C \cdot Q = E(t). \quad (2')$$

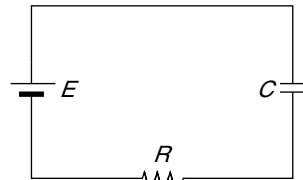


Figure 2

Differentiating the equations with respect to t finds:

$$R \cdot I' + 1/C \cdot I = E'(t). \quad (3)$$

■ Series RLC-Circuit(charge/current)

Voltage drop V_R across the resistor in RLC -circuit (Figure 3) is RI . Voltage drop V_L across the inductor is $L \cdot di/dt$. Voltage drop V_C across the capacitor is $Q/C = 1/C \cdot \int I dt$. According to KVL , the sum of two voltage drops is equivalent to electromotive force $E(t)$.

Thus:

$$L \cdot I' + RI + 1/C \cdot \int I \cdot dt = E(t). \quad (4)$$

Differentiating (4) with respect to t , we obtain (5).

$$L \cdot I'' + RI' + 1/C \cdot I = E'(t). \quad (5)$$

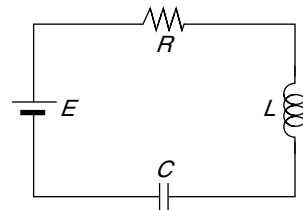


Figure 3



*1Kirchhoff's Voltage law (KVL): The sum of the voltage changes around a closed loop is zero.

■ RL-Circuit

•••••

Example Consider RL-circuit with $L = 2$ henrys, $R = 6$ ohms, and a battery supplying a constant voltage of $E = 12$ volts.

(1) Write a differential equation that relates the valuable current I and time t .

(2) Graph the solution of (1).

$x_0 = 0, y_0 = 0, 0 \leq x \leq 5, h = 0.1$.

Use the following V-Window settings.

Xmin = -1.3, Xmax = 6.3, Xscale = 1,

Ymin = -1, Ymax = 3.1, Yscale = 1.

Answer

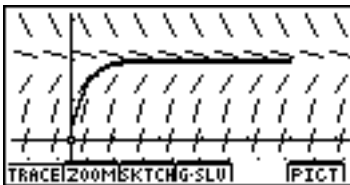
(1) $2 \frac{dI}{dt} + 6I = 12^{*1}$.

(2) See Result Screen.

Procedure

- | | |
|---|---|
| <p>① MENU DIFF EQ</p> <p>② F1 (1st) 4 (Others)</p> <p>③ 6 − 3 ALPHA − (Y) EXE
($y' = 6 - 3y$)</p> <p>④ 0 EXE ($x_0 = 0$) 0 EXE ($y_0 = 0$)</p> <p>⑤ F5 (SET) 1 (Param) F1 (INIT)
0 EXE 5 EXE ($0 \leq x \leq 5$)
0 ▸ 1 EXE ($h = 0.1$) ESC</p> <p>⑥ F5 (SET) 2 (Output) F4 (INIT) ESC
(Select y to graph.)</p> | <p>⑦ SHIFT OPTN (V-Window) F1 (INIT)
(←) 1 ▸ 3 EXE (Xmin = -1.3)
6 ▸ 3 EXE (Xmax = 6.3)
1 EXE (Xscale = 1) ▼
(←) 1 EXE (Ymin = -1)
3 ▸ 1 EXE (Ymax = 3.1)
1 EXE (Yscale = 1) ESC</p> <p>⑧ F6 (CALC)</p> |
|---|---|

Result Screen



• As t increases, the current tends to change to a current of 2 amps.



*¹Think about why this happens! Though your calculator can perform calculations for you, it

cannot take your place when considering the logic behind a calculation.

RC-Circuit

•••••

Example An uncharged capacitor and a resistor are connected in series to a battery as in Figure 2. If $R = 8.00 \times 10^5 \Omega$, $C = 5.00 \mu\text{F}$ ($5.00 \times 10^{-6}\text{F}$), and $E(t) = 12.0 \text{ V}$.

(1) Write down a differential equation that relates current I and time t .

(2) Graph the solution of (1).

$I(0) = 15(\mu\text{A}) = EIR$, $t_0 = 0(\text{sec.})$, $0 \leq t \leq 10$, $h = 0.05$.

Use the following V-Window settings.

$X_{\min} = -1.3$, $X_{\max} = 11.3$, $X_{\text{scale}} = 1$.*

(3) Find the time when the current decrease to e^{-1} of its initial value.

Answer

(1) $8.00 \times 10^5 \cdot I' + 1/(5.00 \times 10^{-6}) \cdot I = 0$.*

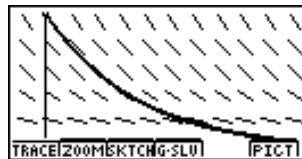
(2) See Result Screen.

(3) 4 seconds.

Procedure

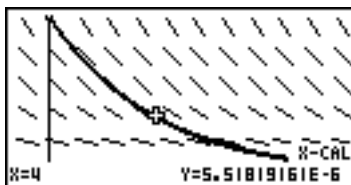
- ① **MENU** **DIFF EQ**
- ② **F1** (1st) **2** (Linear)
- ③ **1** **÷** **(** **ALPHA** **6** (R) **ALPHA** **In** (C) **)**
EXE ($f(x) = 1/(RC)$)
0 **EXE** ($g(x) = 0$)
- ④ **CTRL** **F4** (CAT/CAL)*³
8 **EXP** **5** **→** **ALPHA** **6** (R) **EXE**
 $(8.00 \times 10^5 \rightarrow R)$
5 **EXP** **(-)** **6** **→** **ALPHA** **In** (C) **EXE**
 $(5.00 \times 10^{-6} \rightarrow C)$ **ESC**
- ⑤ **0** **EXE** ($x_0 = 0$)
1 **.** **5** **EXP** **(-)** **5** **EXE** ($y_0 = 15 \times 10^{-6}$)
- ⑥ **F5** (SET) **1** (Param) **F1** (INIT)
0 **EXE** **1** **0** **EXE** ($0 \leq x \leq 10$)
0 **.** **0** **5** **EXE** ($h = 0.05$) **ESC**

- ⑦ **F5** (SET) **2** (Output) **F4** (INIT) **ESC**
 (Select y to graph.)
- ⑧ **SHIFT** **OPTN** (V-Window) **F1** (INIT)
(←) **1** **.** **3** **EXE** ($X_{\min} = -1.3$)
1 **1** **.** **3** **EXE** ($X_{\max} = 11.3$)
1 **EXE** ($X_{\text{scale}} = 1$) **ESC***¹
- ⑨ **F6** (CALC) **F2** (ZOOM) **5** (Auto)*⁴



- ⑩ **F4** (G.SLV) **7** (X-Cal)*⁵
1 **5** **EXP** **(-)** **6** **×** **SHIFT** **In** (e^x) **(←)**
1
EXE (Determines x when $y = 15 \times 10^{-6} \times e^{-1}$.)

Result Screen





*¹Here, Ymin and Ymax are not specified. Adjust Ymin and Ymax after drawing the graph.

*²Think about why this happens! Though your Algebra FX calculator can perform calculations for you, it cannot take your place when considering the logic behind a calculation.

*³While the graph is on the display, press $\boxed{\text{CTRL}} \boxed{\text{F4}}$ to open the Calc Window. After using the Calc Window to change the value of a variable connected with a graph, refresh the formula or initial values. Doing so ensures that the displayed graph is current.

*⁴After graphing the differential equation, perform $\boxed{\text{F4}} \boxed{\text{(ZOOM)}} \boxed{\text{5}} \boxed{\text{(Auto)}}$ to adjust the V-Window y-axis so the entire solution curve fits in the screen along the y-axis.

*⁵ $\boxed{\text{F4}} \boxed{\text{(G-SLV)}} \boxed{\text{7}} \boxed{\text{(X-Cal)}}$ calculates the value of x for any value of y on the graphed solution curve $f(x) = y$.

Also:

$\boxed{\text{F4}} \boxed{\text{(G-SLV)}} \boxed{\text{1}} \boxed{\text{(Root)}}$:Solves for $f(x) = 0$ from a graphed solution curve.

$\boxed{\text{F4}} \boxed{\text{(G-SLV)}} \boxed{\text{2}} \boxed{\text{(Max)}}$:Calculates the local maximum of a graphed solution curve.

$\boxed{\text{F4}} \boxed{\text{(G-SLV)}} \boxed{\text{3}} \boxed{\text{(Min)}}$:Calculates the local minimum of a graphed solution curve.

$\boxed{\text{F4}} \boxed{\text{(G-SLV)}} \boxed{\text{4}} \boxed{\text{(Y-Intpt)}}$:Calculates the Y-intercept of a graphed solution curve.

$\boxed{\text{F4}} \boxed{\text{(G-SLV)}} \boxed{\text{5}} \boxed{\text{(Isect)}}$:Determines the intersects when two or more solution curves are graphed.

$\boxed{\text{F4}} \boxed{\text{(G-SLV)}} \boxed{\text{6}} \boxed{\text{(Y-Cal)}}$:Calculates the value of y for any value of x on the graphed solution curve $f(x) = y$.

9. Physics - Van del Pol Equation

Description

The van del Pol equation

$$d^2i/d^2t - \epsilon(1 - i^2) di/dt + i = 0$$

is a classic source of nonlinear oscillatory phenomena. The first description and analysis of this equation by van del Pol in the mid 1920's was in terms of "negative resistance" in vacuum tube circuits.

• • • • •

Example (a) Express the differential equation below as a set of 1st order differential equations,

$$d^2i/d^2t - E(1 - i^2) di/dt + i = 0. \quad (1)$$

(b) Graph the solution for the linear differential equation in (1), above.

Here, $E = 0, 0.1, 2.0,$

$x_0 = 0, y_0 = 0, y'_0 = 1, -5 \leq x \leq 15, h = 0.1.$

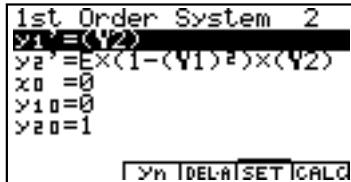
Procedure (a)

- | | | |
|---|---|---|
| ① | MENU DIFF EQ | |
| ② | F3 (N-th) F2 (2)(Order:2) | |
| ③ | ALPHA cos (E) X C 1 = ALPHA = (Y) | ④ 0 EXE ($x_0 = 0$) |
| | x²) X F3 ($y(n)$) 1 = | 0 EXE ($y_0 = 0$) |
| | ALPHA = (Y) EXE | 1 EXE ($y'_0 = 1$) |
| | ($y'' = E \times (1 - y^2)y' - y$) | ⑤ F2 (\rightarrow SYS) EXE (Yes) |

The differential equation is converted to a set of first order differential equations as shown below.

$$y_1' = y_2,$$

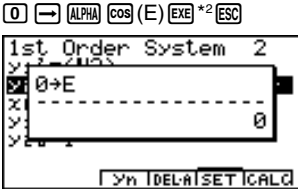
$$y_2' = E \times (1 - (y_1)^2) \times y_2 - y_1.$$



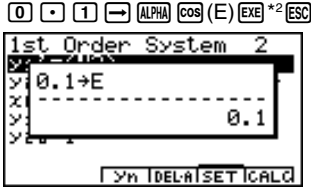
Procedure (b)

- ① Use the formula and initial conditions you input for Procedure (a).
- ② **CTRL** **F4** (CAT/CAL)*1
- ③

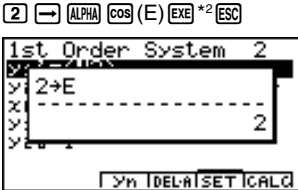
In the case $E = 0$:



In the case $E = 0.1$:



In the case $E = 2.0$:



- ④ **F5** (SET) **1** (Param) **F1** (INIT)
(←) **5** **EXE** **1** **5** **EXE** ($-5 \leq x \leq 15$)
0 **·** **1** **EXE** ($h = 0.1$) **ESC**
- ⑤ **F5** (SET) **2** (Output) **F4** (INIT)
▼ **▼** **F1** (SEL)
 (Select (y_1) and (y_2) to graph.)

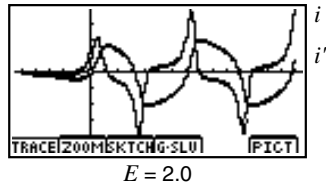
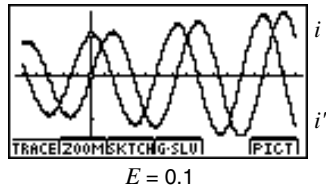
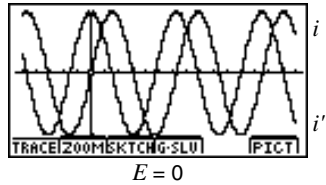
▲ **▲** **F2** (LIST) **1** **EXE**
 (Select LIST1 to store the values for x in LIST1.)

▼ **F2** (LIST) **2** **EXE**
 (Select LIST2 to store the values for (y_1) in LIST2.)

▼ **F2** (LIST) **3** **EXE**
 (Select LIST3 to store the values for (y_2) in LIST3.) **ESC**

- ⑥ **SHIFT** **OPTN** (V-Window) **F1** (INIT)
(←) **5** **·** **5** **EXE** **1** **5** **·** **5** **EXE**
1 **EXE** **ESC** *3

- ⑦ **F6** (CALC) **F2** (Zoom) **5** (Auto)*3



*1 After using the Calc Window to change the value of a variable connected either a graph, re-input one of the initial conditions and redraw the graph. Doing so ensures that the displayed graph or table is current.

*2 While the graph is on the display, press **CTRL** **F4** to open the Calc Window. After using the Calc Window to change the value of a

variable connected with a graph, refresh the formula or initial values. Doing so ensures that the displayed graph is current.

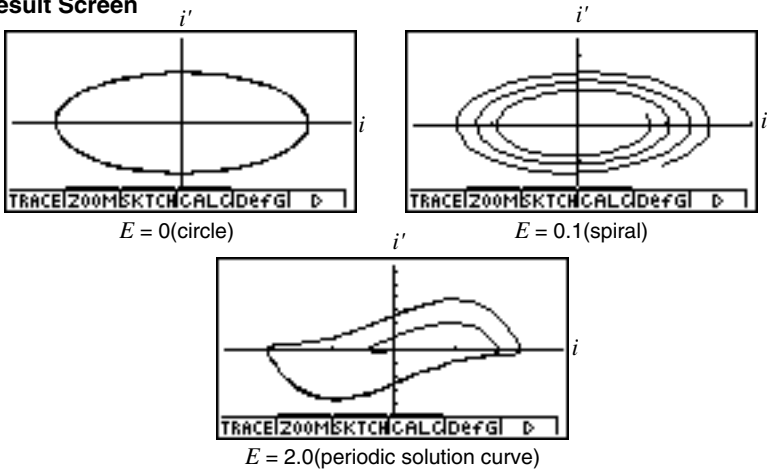
*3 After graphing the differential equation, perform **F2** (ZOOM) **5** (Auto) to adjust the V-Window y-axis so the entire solution curve fits in the screen along the y-axis.

To further analyze the result, we can graph the relation between (i) and (i').

Procedure (continued)

- | | |
|---|--|
| <p>① MENU STAT (List 2 and List 3 contain values for (y_1) and (y_2) respectively.)</p> <p>② F1 (GRPH) 5 (Set)</p> <p>③ F1 (GPH1)</p> <p style="padding-left: 20px;">▼ F2 (xy)</p> <p style="padding-left: 20px;">▼ F1 (LIST) 2 EXE
(XList = List2 : (y_1))</p> | <p>▼ F1 (LIST) 3 EXE
(YList = List3 : (y_2))</p> <p>▼ ▼ F3 (■) ESC</p> <p>CTRL F3 (SET UP) F1 (Auto) ESC</p> <p>④ F1 (GRPH) 1 (S-Gph1)</p> |
|---|--|

Result Screen



Other Things to Do

- Compare the solution curves.



*1 In the STAT Mode, use **CTRL** **F3** (SET UP) to display the SET UP variable, and then use **F2** (Man) to change the StatWind setting to Manual. After you do this, the V-Window

shows the values used for settings made using **SHIFT** **OPTN** (V-Window). You cannot use this screen to adjust Xmin, Xmax, Ymin, and Ymax.

10. Biology - Predator-prey Equations

Parasite species P hatches its eggs in host species H , and the deposit of each egg of P causes the death of a member of H . For each 100 of population:

Let H_b denote the number of births per year of the H species, and H_d denote the number of deaths per year of the H species.

We now make the following assumptions:

1. That H_b , H_d , and P_d are constants. Hence if y_1 is the population of the H species at time x and y_2 is the population of the P species, then in time dx
 - (a) $H_b (y_1/100) dx$ is the approximate number of births of host H ,
 - (b) $H_d (y_1/100) dx$ is the approximate number of natural deaths of host H ,
 - (c) $P_d (y_2/100) dx$ is the approximate number of natural death of parasite P .
2. That the number of eggs per year deposited by the P species resulting in the death of the H species is proportional to the probability that the members of the two species come into contact with each other. Since this probability depends on the product $y_1 y_2$ of the two populations, we assume that in time dx ,
 - (d) $ky_1 y_2 dx$ is the approximate number of deaths of the host due to the presence of parasite P , where k is proportionality constant. Hence, also, since each such death of H implies the laying of an egg by P ,
 - (e) $ky_1 y_2 dx$ is the approximate number of births of P .

Therefore,

$$dy_1 = H_b (y_1/100) dx - H_d (y_1/100) dx - ky_1 y_2 dx,$$

$$dy_2 = ky_1 y_2 dx - P_d (y_2/100) dx$$

that is,

$$(y_1)' = hy_1 - ky_1 y_2 = (h - ky_2) y_1$$

$$(y_2)' = ky_1 y_2 - py_2 = (ky_1 - p) y_2$$

where,

$$h = (H_b - H_d)/100, \quad p = P_d/100.$$

The differential equations form a nonlinear system.

They are called *predator-prey equations*.



Example To graph the solution of the system of first order differential equations below.

$$(y_1)' = (2 - (y_2)) (y_1),$$

$$(y_2)' = (2 (y_1) - 3) (y_2),$$

$$x_0 = 0, (y_1)_0 = 1, (y_2)_0 = 1/4, 0 \leq x \leq 10, h = 0.1.$$

Use the following V-Window settings.

$$Xmin = -1, \quad Xmax = 11, \quad Xscale = 1,$$

$$Ymin = -1, \quad Ymax = 8, \quad Yscale = 1.$$

Procedure

- ① **MENU** DIFF EQ
- ② **F4**(SYS)
- ③ **F2**(2)
- ④ **(** **2** **-** **F3**(y_n) **2** **)** **X** **F3**(y_n)
1 **EXE** ($y_1' = (2 - y_2 \times y_1)$)
(**2** **X** **F3**(y_n) **1** **-** **3**
) **X** **F3**(y_n) **2** **EXE** ($y_2' = (2y_1 - 3)y_2$)
- ⑤ **0** **EXE** ($x_0 = 0$)
1 **EXE** ($(y_1)_0 = 1$)
1 **÷** **4** **EXE** ($(y_2)_0 = 0.25$)
- ⑥ **F5**(SET) **1**(Param) **F1**(INIT)
- ⑦ **0** **EXE** **1** **0** **EXE** ($0 \leq x \leq 10$)
- ⑧ **0** **.** **1** **EXE** ($h = 0.1$)

- ⑨ **ESC** **F5**(SET) **2**(Output) **F4**(INIT)
▼ **▼** **F1**(SEL)
 (Select (y_1) and (y_2) to graph.)
▲ **▲** **F2**(LIST) **1** **EXE** (Select LIST1 to store the values for x in LIST1.)
▼ **F2**(LIST) **2** **EXE** (Select LIST2 to store the values for (y_1) in LIST2.)
▼ **F2**(LIST) **3** **EXE** (Select LIST3 to store the values for (y_2) in LIST3.)

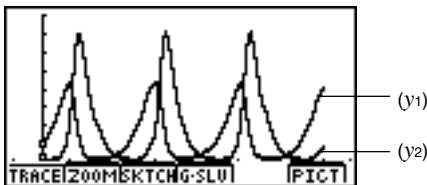
	1st	2nd	3rd	4th	5th
x	1	-	-	-	-
y1	2	-	-	-	-
y2	3	-	-	-	-

SEL LIST Param INIT RCL

Parameter	
Xrangs	
min	:0
max	:10
h	:0.1
STEP	:1
SF	:12
INITI	

- ⑩ **ESC** **SHIFT** **OPTN** (V-Window)
(←) **1** **EXE** **1** **1** **EXE** **1** **EXE** **▼**
(←) **1** **EXE** **8** **EXE** **1** **EXE** **ESC**
- ⑪ **F6**(CALC)

Result Screen



This is a periodic solution.

11. Chaos - Strange Attractor

Let's try drawing the "chaotic" differential equation trajectory (attractor).

■ Lorenz Attractor

In 1963, meteorologist Edward Lorenz created a meteorological model using the following system of simultaneous differential equations,

$$y_1' = -\sigma \cdot y_1 + \sigma \cdot y_2,$$

$$y_2' = R \cdot y_1 - y_2 + y_1 \cdot y_3,$$

$$y_3' = -B \cdot y_3 + y_1 \cdot y_2,$$

where σ , R , B are constants.

Here, y_1 , y_2 , and y_3 are functions of time x . If you try to obtain a numerical solution for this system of differential equations, you will find out that even a slight difference in initial conditions produces a completely different solution curve.

■ Rössler Attractor

In 1976 Otto E. Rössler found a simple system. His system of differential equations is

$$y_1' = -y_2 - y_3,$$

$$y_2' = y_1 + a \cdot y_2,$$

$$y_3' = b + y_1 \cdot y_3 - c \cdot y_3,$$

where a , b , c are constants. This is an artificial system designed with the purpose of creating model for a strange attractor.

■ Lorenz Attractor



Example To graph the solution of the system of differential equations

$$y_1' = -3(y_1 - y_2),$$

$$y_2' = -y_1y_3 + 28y_1 - y_2,$$

$$y_3' = y_1y_2 - y_3,$$

$$y_1(0) = 0, y_2(0) = 1, y_3(0) = 0,$$

$0 \leq x \leq 10$, $h = 0.025$ and **Step = 2**.

Use the following V-Window settings,

Xmin = -1.3, **Xmax = 11.3**.

Procedure

- ① **MENU** DIFF EQ
- ② **F4** (SYS) **F3** (n) **3** **EXE** ($n = 3$)
- ③ **(←)** *¹ **3** **(←)** **F3** (y_n) **1** **(-)** **F3** (y_n)
2 **(→)** **EXE** ($y_1' = -3(y_1 - y_2)$),
(←) **F3** (y_n) **1** **F3** (y_n) **3** **+** **2** **8**
F3 (y_n) **1** **(-)** **F3** (y_n) **2** **EXE**
 $(y_2' = -y_1y_3 + 28y_1 - y_2)$,
F3 (y_n) **1** **F3** (y_n) **2** **(-)** **F3** (y_n) **3**
EXE ($y_3' = y_1y_2 - y_3$)
- ④ **0** **EXE** ($x_0 = 0$) **0** **EXE** ($(y_1)_0 = 0$)
1 **EXE** ($(y_2)_0 = 1$) **0** **EXE** ($(y_3)_0 = 0$)
- ⑤ **F5** (SET) **1** (Param) **F1** (INIT)
0 **EXE** **1** **0** **EXE** ($0 \leq x \leq 10$)
0 **(.)** **0** **2** **5** **EXE** ($h = 0.025$)*²
2 **EXE** (Step = 2)*³ **ESC**
- ⑥ **F5** (SET) **2** (Output) **F4** (INIT)
(↓) **(↓)** **F1** (SEL) **(↓)** **F1** (SEL)
 (Select (y_1) , (y_2) and (y_3) to graph)
(↑) **(↑)** **F2** (LIST) **1** **EXE**
 (Select LIST1 to store the values for (y_1) in LIST1.)
(↓) **F2** (LIST) **2** **EXE**
 (Select LIST2 to store the values for (y_2) in LIST2.)
(↓) **F2** (LIST) **3** **EXE**
 (Select LIST3 to store the values for (y_3) in LIST3.) **ESC**
- ⑦ **SHIFT** **(OPTN)** (V-Window) **F1** (INIT)
(←) **1** **(.)** **3** **EXE** (Xmin = -1.3)
1 **1** **(.)** **3** **EXE** (Xmax = 11.3) **ESC***⁴
- ⑧ **F6** (CALC) **F2** (ZOOM) **5** (Auto)*⁵



*¹Do not confuse the **(-)** key and the **(←)** key. An error occurs if you use the **(←)** key as the subtraction symbol.

*²An error occurs in the DIFF EQ Mode when $((Xrange\ max) - (Xrange\ min)) / (h) \times (Step) > 254$. An error occurs when $h = 0.025$, $0 \leq x \leq 10$ and Step=1.

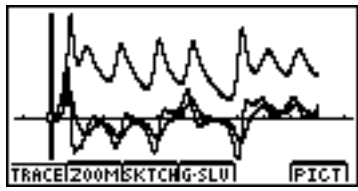
*³When graphed for the first time, a function is always graphed with every step. When the

function is graphed again, however it is graphed according to a value of the step.

*⁴Here, Ymin and Ymax are not specified. Adjust Ymin and Ymax after drawing the graph.

*⁵After graphing the differential equation, perform **F2** (ZOOM) **5** (Auto) to adjust the V-Window y-axis so the entire solution curve fits in the screen along the y-axis.

Result Screen



Now you can graph the relationship between variables y_1 , y_2 , and y_3 based on the results we calculated here.

Procedure

① **MENU** STAT (List 1, List 2, and List 3 contain values for (y_1) , (y_2) , and (y_3) respectively.)

▼ **F1** (LIST) **3** **EXE**
(YList = List3 : (y_3))

② **F1** (GRPH) **5** (Set)
F1 (GPH1) ▼ **F2** (xy)

In the case of (c) :

▼ **F1** (LIST) **2** **EXE**
(XList = List2 : (y_2))

In the case of (a) :

▼ **F1** (LIST) **1** **EXE**
(XList = List1 : (y_1) *1)

▼ **F1** (LIST) **3** **EXE**
(YList = List3 : (y_3))

▼ **F1** (LIST) **2** **EXE**
(YList = List2 : (y_2) *1)

▼ ▼ **F3** (■) **ESC**

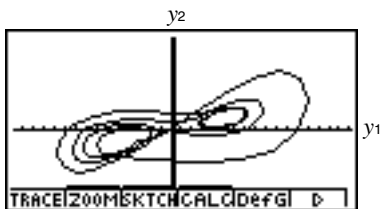
In the case of (b) :

▼ **F1** (LIST) **1** **EXE**
(XList = List1 : (y_1))

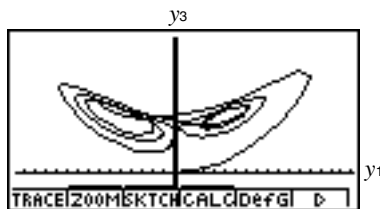
③ **CTRL** **F3** (SET UP) **F1** (Auto) **ESC**

④ **F1** (GRPH) **1** (S-Gph1)

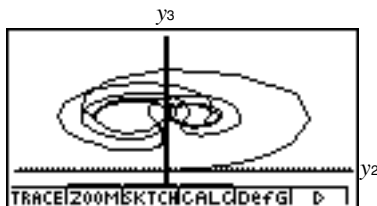
Result Screen



(a) Top view of the Lorenz attractor



(b) Side view of the Lorenz attractor



(c) Side view of the Lorenz attractor



*1 Here, LIST1 is set for y_1 , LIST2 for y_2 , and LIST3 for y_3 .

*2 In the STAT Mode, use **CTRL** **F3** (SET UP) to display the SET UP variable, and then use **F2** (Man) to change the StatWind setting to

Manual. After you do this, the V-Window shows the values used for settings made using **SHIFT** **OPTN** (V-Window). You cannot use this screen to adjust Xmin, Xmax, Ymin, and Ymax.

■ Rössler Attractor



Example To graph the solution of the system of differential equations

$$y_1' = -y_2 - y_3, \quad y_2' = y_1 + 0.2y_2, \quad y_3' = 0.2 - (7 - y_1)y_3,$$

$$y_1(0) = 5, \quad y_2(0) = 5, \quad y_3(0) = 7.$$

$$0 \leq x \leq 50, \quad h = 0.2 \text{ (Step = 1)}.$$

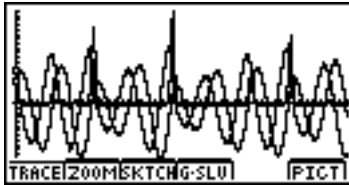
Use the following V-Window settings.

$$Xmin = -0.5, \quad Xmax = 50.5.^{*1}$$

Procedure

- ① **MENU** (DIFF EQ)
- ② **F4** (SYS) **F3** (n) **3** **EXE** **F4** (DEL·A) **EXE**
- ③ **(-)** \times **F3** (y_n) **2** **=** **F3** (y_n) **3** **EXE**
 $(y_1' = -y_2 - y_3)$
F3 (y_n) **1** **+** **0** \cdot **2** **F3** (y_n) **2** **EXE**
 $(y_2' = y_1 + 0.2y_2)$
0 \cdot **2** **=** **(** **7** **=** **F3** (y_n) **1** **)**
F3 (y_n) **3** **EXE**
 $(y_3' = 0.2 - (7 - y_1)y_3)$
- ④ **0** **EXE** ($x_0 = 0$) **5** **EXE** ($(y_1)_0 = 5$)
5 **EXE** ($(y_2)_0 = 5$) **7** **EXE** ($(y_3)_0 = 7$)
- ⑤ **F5** (SET) **1** (Param) **F1** (INIT)
0 **EXE** **5** **0** **EXE** ($0 \leq x \leq 50$)
0 \cdot **2** **EXE** ($h = 0.2$) **ESC**
- ⑥ **F5** (SET) **2** (Output) **F4** (INIT)
(v) **(v)** **F1** (SEL) **(v)** **F1** (SEL)
 (Select (y_1) , (y_2) and (y_3) to graph)
(v) **(v)** **F2** (LIST) **1** **EXE** (Select LIST1
 to store the values for (y_1) in LIST1.)
(v) **F2** (LIST) **2** **EXE** (Select LIST2
 to store the values for (y_2) in LIST2.)
(v) **F2** (LIST) **3** **EXE** (Select LIST3
 to store the values for (y_3) in
 LIST3.) **ESC**
- ⑦ **SHIFT** (OPTN) (V-Window) **F1** (INIT)
(-) **0** \cdot **5** **EXE** ($Xmin = -0.5$)
5 **0** \cdot **5** **EXE** ($Xmax = 50.5$) *1 **ESC**
- ⑧ **F6** (CALC) **F2** (ZOOM) **5** (Auto) *3

Result Screen



*1 Here, $Ymin$ and $Ymax$ are not specified. Adjust $Ymin$ and $Ymax$ after drawing the graph.

*2 Do not confuse the **=** key and the **(-)** key. An error occurs if you use the **(-)** key as the subtraction symbol.

*3 After graphing the differential equation, perform **F2** (ZOOM) **5** (Auto) to adjust the V-Window y -axis so the entire solution curve fits in the screen along the y -axis.

Now you can graph the relationship between variables y_1 , y_2 , and y_3 based on the results we calculated here.

Procedure

- ① **MENU** STAT (List 1, List 2, and List 3 contain values for (y_1) , (y_2) , and (y_3) respectively.)

▼ **F1** (LIST) **3** **EXE**
(YList = List3 : (y_3))

- ② **F1** (GRPH) **5** (Set)
F1 (GPH1) ▼ **F2** (xy)

In the case of (c) :

▼ **F1** (LIST) **2** **EXE**
(XList = List2 : (y_2))

In the case of (a) :

▼ **F1** (LIST) **1** **EXE**
(XList = List1 : (y_1))

▼ **F1** (LIST) **3** **EXE**
(YList = List3 : (y_3))

▼ **F1** (LIST) **2** **EXE**
(YList = List2 : (y_2))

▼ ▼ **F3** (■) **ESC**

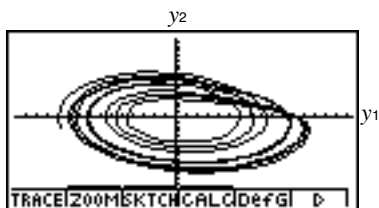
- ③ **CTRL** **F3** (SET UP) **F1** (Auto) **ESC**

In the case of (b) :

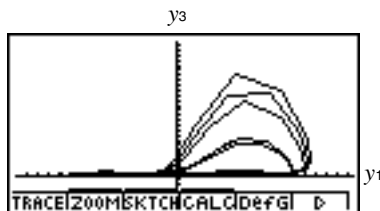
▼ **F1** (LIST) **1** **EXE**
(XList = List1 : (y_1))

- ④ **F1** (GRPH) **1** (S-Gph1)

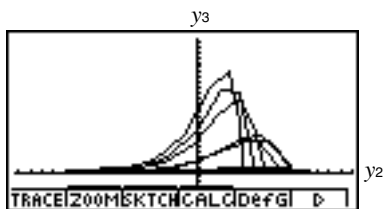
Result Screen



(a) Top view of the Rössler attractor



(b) Side view of the Rössler attractor



(c) Side view of the Rössler attractor

12. Chemistry - The 'Brusselator'

The Brusselator is a model for an oscillating chemical reaction proposed by Prigogine,

$$y_1' = a - by_1 + y_1^2 y_2 - y_1,$$

$$y_2' = by_1 - y_1^2 y_2.$$

y_1 and y_2 are always positive for a given $x > 0$. a and b are positive constants.

Chemical Equilibrium

In the case of $b < 1 + a^2$, the solution (y_1, y_2) converges on $(a, b/a)$.

Oscillating Reaction

A periodic solution exists in the case of $b > 1 + a^2$.

When initial values $(y_1(0), y_2(0))$ are anything other than $(a, b/a)$, solution (y_1, y_2) approaches the periodic solution.



Example To graph the solution of the system of differential equations

$$y_1' = 1 - 4y_1 + y_1^2 y_2, \quad y_2' = 3y_1 - y_1^2 y_2,$$

$$y_1(0) = 0, \quad y_2(0) = 1.$$

$$0 \leq x \leq 40, \quad h = 0.05 \text{ and Step} = 4.$$

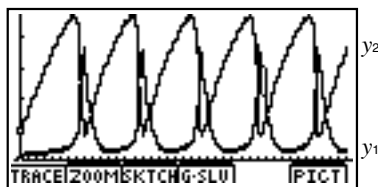
Use the following V-Window settings.

$$\text{Xmin} = -0.5, \quad \text{Xmax} = 40.5.$$

Procedure

- ① **MENU** DIFF EQ
- ② **F4** (SYS) **F2** (2)
- ③ **1** **=** *¹ **4** **F3** (y_n) **1** **+** **F3** (y_n) **1** **x²**
F3 (y_n) **2** **EXE** ($y_1' = 1 - 4y_1 + y_1^2 y_2$)
3 **F3** (y_n) **1** **=** **F3** (y_n) **1** **x²**
F3 (y_n) **2** **EXE** ($y_2' = 3y_1 - y_1^2 y_2$)
0 **EXE** ($x_0 = 0$) **0** **EXE** ($(y_1)_0 = 0$)
1 **EXE** ($(y_2)_0 = 1$)
- ④ **F5** (SET) **1** (Param) **F1** (INIT)
0 **EXE** **4** **0** **EXE** ($0 \leq x \leq 40$)
0 **.** **0** **5** **EXE** ($h = 0.05$)
4 **EXE** (Step = 4)^{*2*} **ESC**
- ⑤ **F5** (SET) **2** (Output) **F4** (INIT)
▼ **▼** **F1** (SEL) (Select (y_1) and (y_2) to graph.)
▲ **F2** (LIST) **1** **EXE** (Select List1 to store the values for (y_1) in LIST1.)
▼ **F2** (LIST) **2** **EXE** (Select List2 to store the values for (y_2) in LIST2.) **ESC**
- ⑥ **SHIFT** **OPTN** (V-Window) **F1** (INIT)
↵ **0** **.** **5** **EXE** (Xmin = -0.5)
4 **0** **.** **5** **EXE** (Xmax = 40.5) **ESC**^{*4}
- ⑦ **F6** (CALC)
F2 (ZOOM) **5** (Auto)^{*5}

Result Screen



^{*1} Do not confuse the **=** key and the **↵** key. An error occurs if you use the **↵** key as the subtraction symbol.

^{*2} When graphed for the first time, a function is always graphed with every step. When the function is graphed again, however it is graphed according to a value of the step.

^{*3} An error occurs in the DIFF EQ Mode when $((\text{Xrange max}) - (\text{Xrange min})) / (h) \times (\text{Step}) > 254$. For example, an error occurs when $h = 0.05$, $0 \leq x \leq 50$ and Step = 1.

^{*4} Here, Ymin and Ymax are not specified. Adjust Ymin and Ymax after drawing the graph.

^{*5} After graphing the differential equation, perform **F2** (ZOOM) **5** (Auto) to adjust the V-Window y -axis so the entire solution curve fits in the screen along the y -axis.

● Numerical Methods of Differential Equation (RECUR Mode)

■ Euler Method

The RECUR Mode lets you solve 1st order differential equations numerically using the Euler method. The general procedure for solving such equations is shown below.

$$y' = f(x,y), f(x_0) = y_0,$$

$$x_{n+1} = x_n + h,$$

$$y_{n+1} = y_n + f(x_n, y_n) \times h.$$

$$n = 0, 1, 2, \dots$$

■ Heun Method

The RECUR Mode lets you solve 1st order differential equations numerically using the Heun method. The general procedure for solving such equations is below.

$$y' = f(x,y), f(x_0) = y_0,$$

$$x_{n+1} = x_n + h,$$

$$y_{n+1} = y_n(k_1 + k_2)/2,$$

$$k_1 = f(x_n, y_n) \times h,$$

$$k_2 = f(x_{n+1}, y_n + k_1) \times h.$$

$$n = 0, 1, 2, \dots$$

■ Euler method



Example 1 Apply the Euler method to the following initial value problem, choosing $h = 0.2$ and computing y_1, y_2, \dots, y_5 :

$$y' = x + y, x_0 = 0, y_0 = 1.$$

Answer

Here $y' = x + y$, so the initial value problem becomes

$$x_{n+1} = x_n + 0.2,$$

$$y_{n+1} = y_n + (x_n + y_n) \times 0.2,$$

$$x_0 = 0, y_0 = 1.$$

n	x_n	y_n
0	0.0	1.00000
1	0.2	1.20000
2	0.4	1.48000
3	0.6	1.85600
4	0.8	2.34720
5	1.0	2.97664

Procedure

- ① **MENU** **RECUR**
- ② **F3** (TYPE) **2** ($a_{n+1} =$)
F4 ($n.a.n.$) **2** (a_n) **+** **0** **.** **2** **EXE**
 $(a_{n+1} = a_n + 0.2)$
F4 ($n.a.n.$) **3** (b_n) **+** **⏏** **F2** (a_n) **+**
F3 (b_n) **⏏** **×** **0** **.** **2** **EXE**
 $(b_{n+1} = b_n + (a_n + b_n) \times 0.2)$
- ③ **F5** (RANG) **F1** (a_0)
0 **EXE** (Start = 0)
5 **EXE** (End = 5)
0 **EXE** ($a_0 = 0$)
1 **EXE** ($b_0 = 1$) **ESC**
- ④ **F6** (TABL)

Result Screen

$n+1$	a_{n+1}	b_{n+1}
0	0	1
1	0.2	1.2
2	0.4	1.48
3	0.6	1.856

⏏

RE-T|DEL|A| |WEB|G-COM|G-PLT|

$n+1$	a_{n+1}	b_{n+1}
2	0.4	1.48
3	0.6	1.856
4	0.8	2.3472
5	1	2.97664

5

RE-T|DEL|A| |WEB|G-COM|G-PLT|

Other Things to Do

- Find the exact solution and determine the error in your numerical approximation.



Example 2 Apply the Euler method to the following initial value problem, choosing $h = 0.05$ and computing y_1, y_2, \dots, y_{40} :

$$y' = -40x, \quad x_0 = 0, \quad y_0 = 1.$$

Answer

Here $y' = x + y$, so the initial value problem becomes

$$x_{n+1} = x_n + 0.05,$$

$$y_{n+1} = y_n + (-40y_n) \times 0.05,$$

$$x_0 = 0, \quad y_0 = 1.$$

n	x_n	y_n
0	0.0	1.00000
1	0.05	-1.00000
Even number	$0.05n$	1.00000
Odd number	$0.05n$	-1.00000

Procedure

- ① **MENU** **RECUR**
- ② **F3** (TYPE) **2** ($a_{n+1} =$)
 F4 ($n.An\cdot$) **2** (a_n) **+** **0** **.** **0** **5** **EXE**
 $(a_{n+1} = a_n + 0.05)$
 F4 ($n.An\cdot$) **3** (b_n) **+** **(** **(-)** **4** **0**
 F3 (b_n) **)** **X** **0** **.** **0** **5** **EXE**
 $(b_{n+1} = b_n + (-40b_n) \times 0.05)$
- ③ **F5** (RANG) **F1** (a_0)
0 **EXE** (Start = 0)
4 **0** **EXE** (End = 40)
0 **EXE** ($a_0 = 0$)
1 **EXE** ($b_0 = 1$) **ESC**
- ④ **F6** (TABL)

Result Screen

Other Things to Do

- Find the exact solution and determine the error in your numerical approximation. Think about why this phenomenon occurs.
- Try changing the coefficients to $y' = -30x$, $y' = -20x$, etc.

Heun Method



Example Apply the Heun method to the following initial value problem, choosing $h = 0.2$ and computing y_1, y_2, \dots, y_5 :

$$y' = y, x_0 = 0, y_0 = 1.$$

Answer

Here $y' = x$, so the initial value problem becomes

$$x_{n+1} = x_n + h,$$

$$k_1 = f(x_n, y_n) \times h = y_n \times h,$$

$$k_2 = f(x_{n+1}, y_n + k_1) \times h = (y_n + k_1) \times h,$$

$$y_{n+1} = y_n + (k_1 + k_2)/2$$

$$= y_n + (y_n \times h + (y_n + y_n \times h) \times h)/2$$

$$= y_n (1 + h + h^2/2) = y_n (1 + 0.2 + 0.2^2/2),$$

$$x_0 = 0, y_0 = 1.$$

n	x_n	y_n
0	0.0	1.0
1	0.2	1.22000
2	0.4	1.48840
3	0.6	1.81584
4	0.8	2.21533456
5	1.0	2.702708163

Procedure

① **MENU** RECUR

② **F3** (TYPE) **2** ($a_{n+1} =$)

F4 ($n.a_{n-}$) **2** (a_n) **+** **0** **.** **2** **EXE**

($a_{n+1} = a_n + 0.2$)

F4 ($n.a_{n-}$) **3** (b_n) **(** **1** **+**

0 **.** **2** **+** **0** **.** **2** **x²**

+ **2** **)** **EXE**

($b_{n+1} = b_n(1 + 0.2 + 0.2^2/2)$)

③ **F5** (RANG) **F1** (a_0)

0 **EXE** (Start = 0)

5 **EXE** (End = 5)

0 **EXE** ($a_0 = 0$)

1 **EXE** ($b_0 = 1$) **ESC**

④ **F6** (TABL)

Result Screen

$n+1$	x_{n+1}	y_{n+1}
0	0	1
1	0.2	1.22
2	0.4	1.4884
3	0.6	1.8158

$n+1$	x_{n+1}	y_{n+1}
2	0.4	1.4884
3	0.6	1.8158
4	0.8	2.2153
5	1.0	2.702708163

Other Things to Do

- Find the exact solution and determine the error in your numerical approximation.