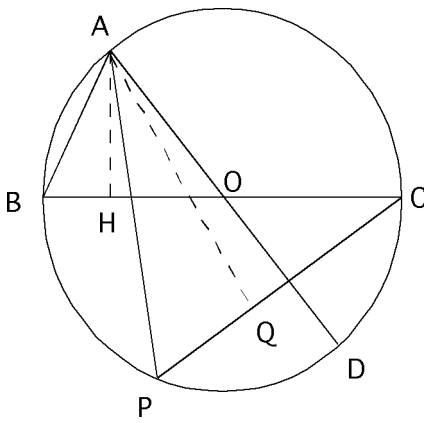
SECRET OF A RIGHT TRIANGLE INSCRIBED IN A CIRCLE

– A right triangle with 30°, 60°, and 90° –

PROBLEM

A triangle ABC is inscribed in a circle of diameter BC and its center O. The segment AH is a line from the point A on the circumference of the circle that falls perpendicular to the diameter BC. The segment AD is another diameter. Let P be a point on the arc BD and draw the segments PA and PC. The segment AQ is a line from the point A that falls perpendicular to the segment PC.
What is the length of the locus of the point Q, when AB = 6 cm, $\angle$ ABC = 60°, and the point P moves on the arc BD from B to D?



The key to this problem is whether one can find the fact that the segment AQ is always perpendicular to the segment PC wherever the point P may be on the arc BD.

1) Wherever the point P may be on the arc BD, $\angle$ AQC = 90° proves true. Therefore the point Q moves on the circumference of a circle of diameter AC. (Theorem on angles at the circumference and the center)

2) The triangle OAB is an equilateral triangle because $\angle$ ABC = 60°, and OA = OB. Therefore, we can find that the angle subtended at the center by the arc BD is 120°, and the point Q moves along the arc of the circle of diameter AC, corresponding to 120°.

3) Because $\angle$ BAC = 90°, AC = $6\sqrt{3}$ cm, we obtain the solution as follows.
120 / 360 x 2 π x $3\sqrt{3}$ = $2\sqrt{3}\,\pi$ cm.

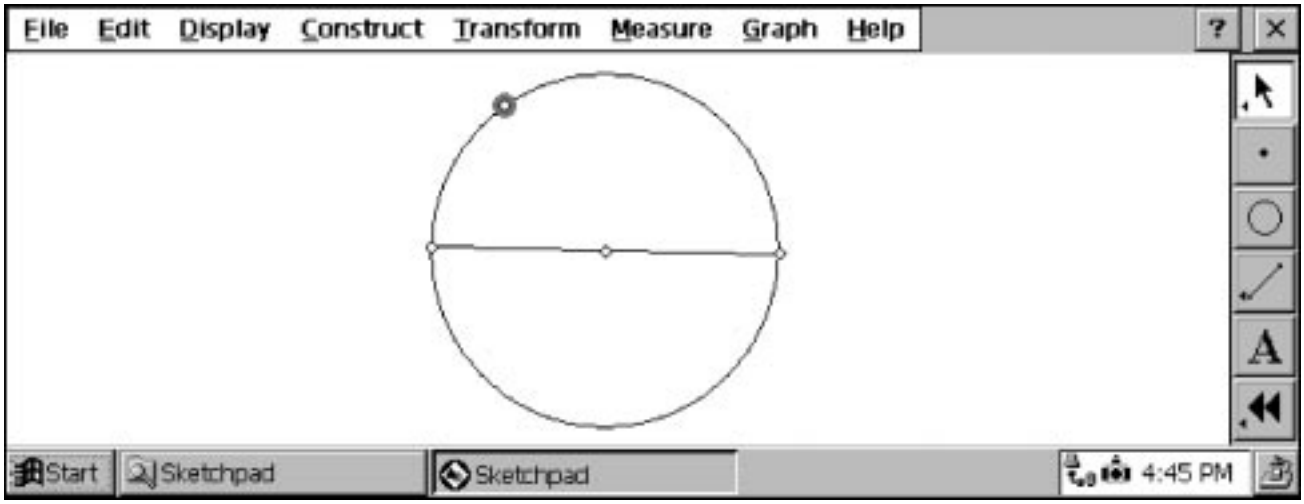
[Let's construct the figure of the problem using the Geometer's Sketchpad]

1) Construction of a circle with the circle palette.

Let B be a point on the circumference. A rotation in space of 180° about the center of the circle maps the point B to the point C, so let's apply the Mark Center command to the center and construct the point C using the Transform-Rotate command.

Draw the segment BC using the Construct-Straight-Object-Segment command.

Specify the circle and construct the point A using the Construct-Point on Circle command.

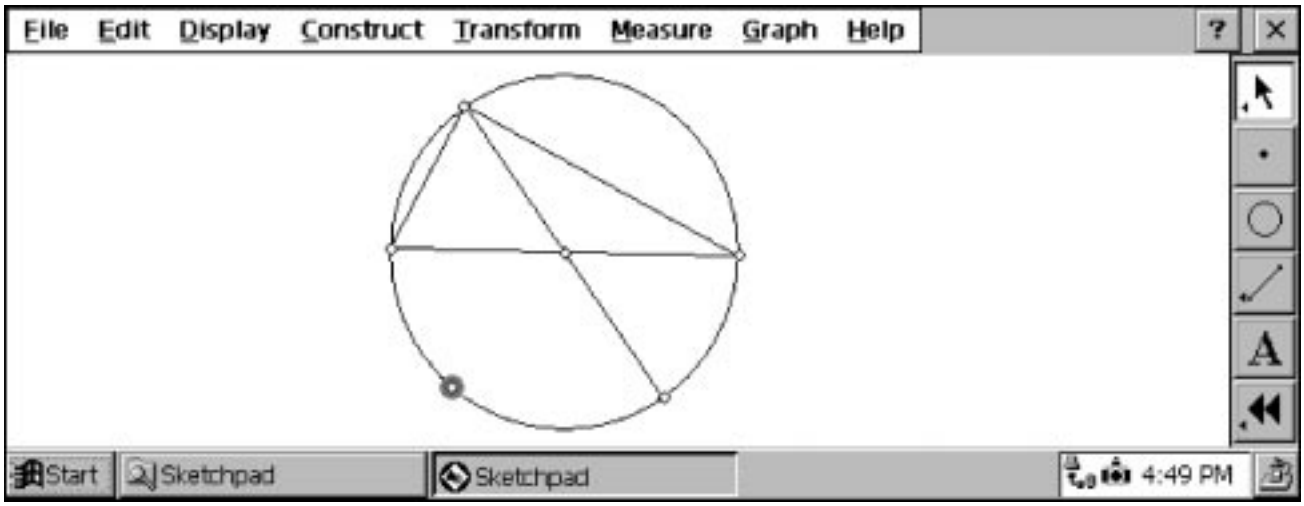


Draw the segments AB and AC.

A rotation in space of 180° about the center O of the circle maps the point A to the point D. Therefore construct the point D using the Rotate command.

Draw the segment AD.

Construct an arbitrary point on the arc BD.



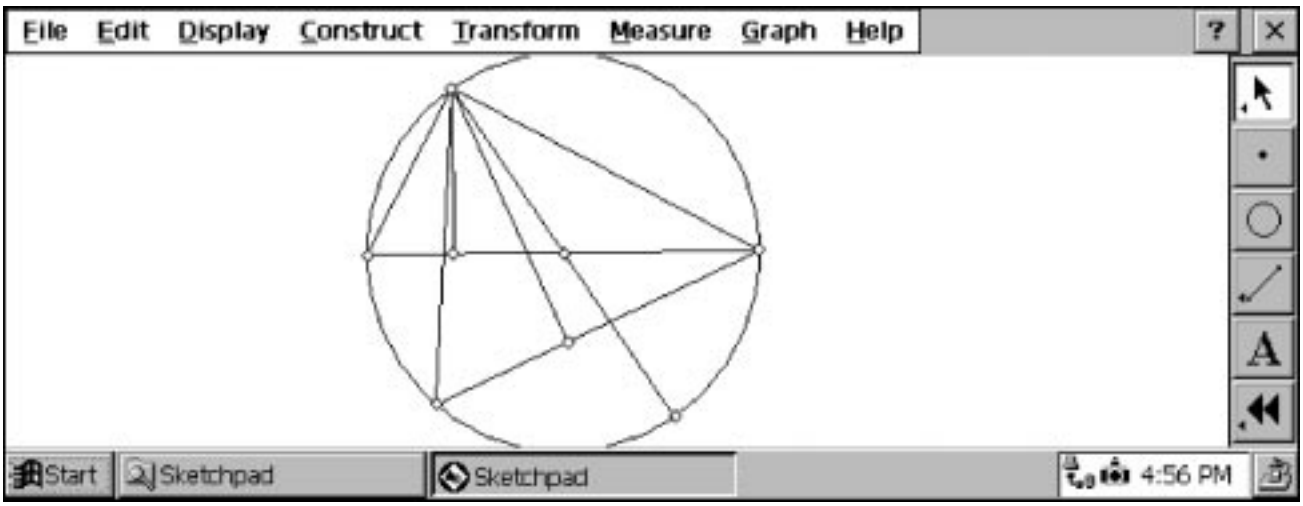
Then draw the segment AH by specifying the segment BC and the point A and performing the Construct-Perpendicular Line command.

Specify a line perpendicular to the segment BC, and construct the point H using the Construct-Intersection command.

Delete this perpendicular using the Display-Hide command and draw the perpendicular AH.

The perpendicular AQ can be drawn in the same way.

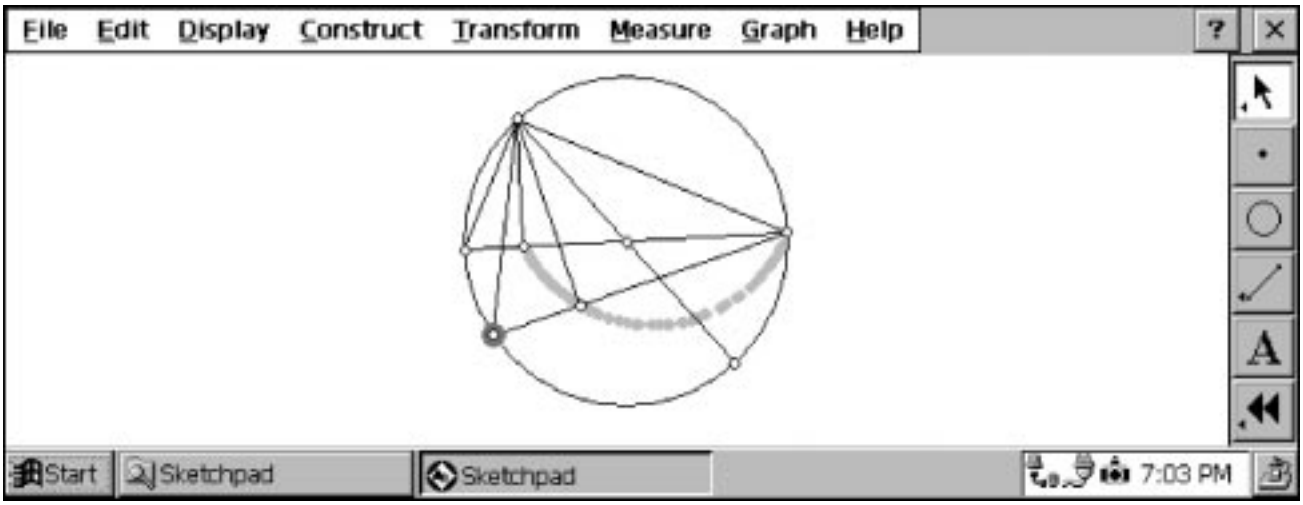
Now we can pinch and move the point P.



[Let's move the figure]

Select the point Q and perform the Trace Intersection command in the Display menu.

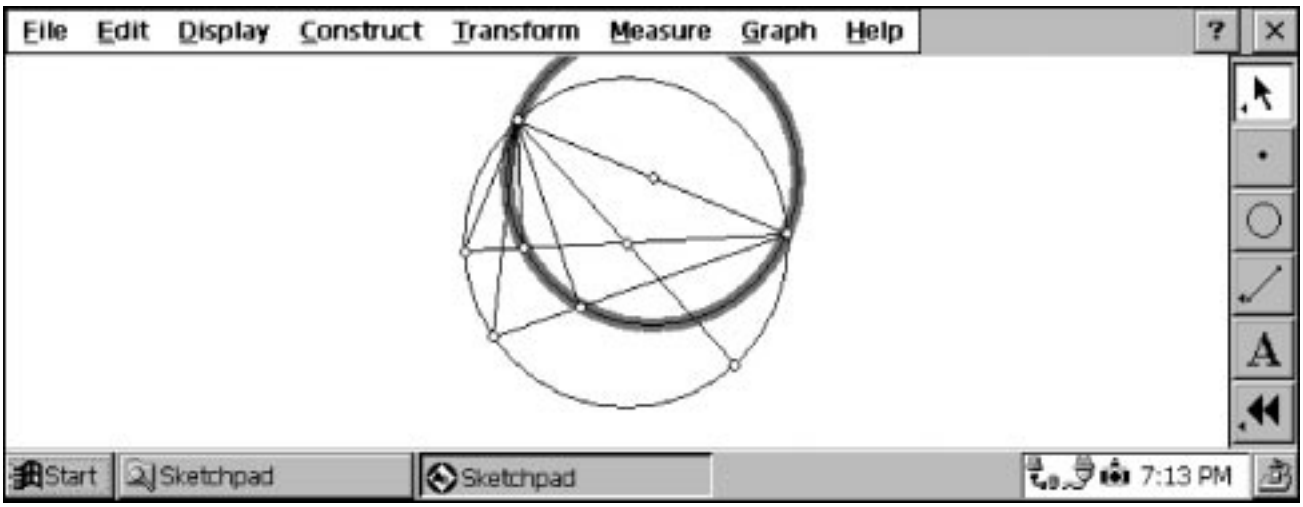
Pinch the point P and move it along the arc BD so that the locus of the point Q is displayed on the screen.



[Research]

Let’s try to confirm that the locus of the point Q is on the circumference of the circle of diameter AC

Construct the midpoint of the segment AC and construct the circle of diameter AC with its center being this midpoint as follows.



We can conclude that the locus of the point Q is on the circumference of the circle of diameter AC.