Size of a mirror to reflect your face
— When a mirror is tilted —

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1. Problem

Mirrors are easy to use in an experiment and provide many mysterious phenomena. Since we can consider phenomena relating to mirrors only with the help of the elementary geometry, they are preferable mathematical materials for junior high school students. Here, we are going to solve a problem that “What is the minimum size of a mirror in which you can see the image of your entire face?” (referred to as “Mirror problem” hereinafter) (Shimada, 1990; Hon, 2000) using the Sketchpad. Particularly, we focus on how the size depends on the slant angle, distance to a mirror, and so forth.

2. Role of the Sketchpad

When we consider the Mirror problem with a schematic to analyze the phenomena, the Sketchpad provides the following features and advantages.

(1) “How to construct a figure” will be the objective of the class and should be accomplished by the students. The ability to construct a figure is required for the students. Once a figure is constructed, we can investigate it both dynamically and quantitatively. As a result, a structure in which “To put the problem into a mathematical form is human, to process the result is technology” is established.

(2) When students, who have not learned trigonometric functions yet, handle the Mirror problem, they can consider only the case in which the mirror and the face are parallel. If, however, the Sketchpad is available, even the junior high school students can consider other cases by constructing a figure. In addition, they can investigate the figure dynamically and consider:

(a) whether or not the size of the mirror depends on the distance when tilted, and how is the dependency, if any; and

(b) whether or not the required size of the mirror depends on the angle when tilted, and how is the dependency, if any

3. Consideration with simple premises

When we consider the problem, “what size is required for a mirror in which you see the image of your entire face?”, the quickest way is to conduct an experiment. While two students collaborate in conducting the experiment, they can find the required size of a mirror by marking the mirror. In addition, it can be expected that the students have questions or feel the necessity to establish conditions for the experiment prior to or during the experiment. For example, the following viewpoints may be brought up.

(a) Does the size depend on the distance?

(b) Is it OK to tilt the mirror?

(c) Does the size of face mean the vertical size or traverse size?

(d) Do we use one eye or both when wee see the mirror?

(e) Is the mirror plane or spherical?

Regarding (a), the students are advised to conduct the experiment while changing the distance because it is a plain question. Regarding (b) to (e), they are advised to consider with simpler premises while taking the possibility to realize into account when they establish the premises. Let's establish the premises as follows here.

For (b), the mirror is placed so that it is parallel to the face. This will make it easier to solve the problem mathematically.

For (c), both vertical and traverse sizes of face are measured, taking the possibility to realize into account.

For (d), we use both eyes according to the conventional manner.

For (e), we use a plane mirror.

With these held in mind of every student, they conduct the experiment in each group. The following results will be provided.

– It seems that the size of the mirror does not depend on the distance.
– The vertical length of the mirror is about half the vertical length of face.
– The traverse length cannot be fixed.

Let's consider the problems, "Is the size of the mirror not influenced by the distance in fact?", and "Is the vertical length of the mirror in fact half the vertical length of face?" Here we use the Sketchpad. It is essential to construct a figure that will explicitly illustrate the phenomena in which a person sees the image of his/her face in the mirror. Before constructing a figure, the following premises should be accepted.

(i) We can see the images of our own faces because the light emitted from the top of the head and from the tip of the jaw is reflected by the mirror and it reached our eyes.

(ii) When light is reflected by the mirror, the incident angle and the angle of reflection are identical.

Taking the abovementioned two points into account, we can construct a figure in the following procedure. It is necessary, however, to note that the notions of face and eye are abstracted and represented by lines and points, respectively.

Step 1

Construct a segment AB (representing the vertical length of face), and a point C (representing an eye) on segment AB.

Step 2

Construct a line l (representing the mirror) parallel to segment AB.

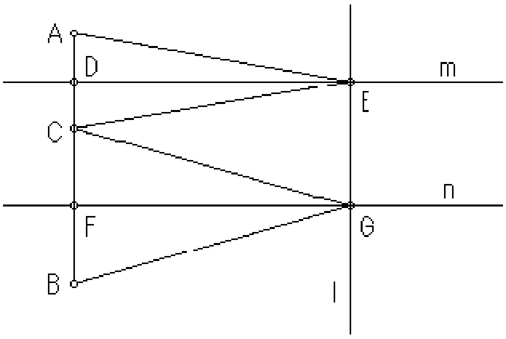
Here, a problem comes up that the incident light and the reflected light should be constructed so that the relation between the incident angle and the angle of reflection holds. This problem relates closely to the proof of the proposition that the size of the mirror is half the size of face, and this procedure is opposite to that of proof. That is, the proof is made using the fact that the incident angle is equal to the angle of reflection, and the construction is, on the contrary, is made so that the incident angle is equal to the angle of reflection. It may be worth trying to find which one is easier. The procedure to construct the figure is as follows.

Step 3

Construct a middle point D on segment AC, a perpendicular m to line l, and let the crossing of two lines be E. Construct lines connecting A and E, and C and E.

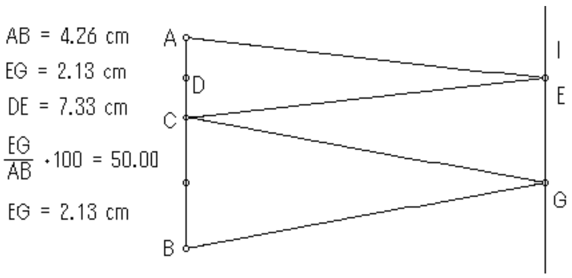
Step 4

Construct a middle point F on segment CB, a perpendicular n to line l, and let the crossing of two lines be G. Construct lines connecting C and G, and B and G.



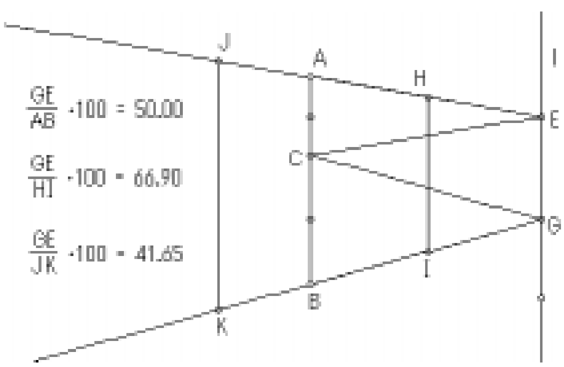
(Display 1)

Measure the length of segment AB, which corresponds to the length of the face, and the length of segment EG, which corresponds to the size required for the mirror. With the help of the calculator features, we can calculate EG/AB, and get the result that the ratio of the size required for the mirror to the size of the face is 50 %. Regarding the problem of the distance dependency, it is advisable to move line l dynamically, which represents the mirror. It will be found out that the ratio of the size required for the mirror to the face size is fixed to 50 % in whichever way you may move line l, as shown in Display 2.



(Display 2)

From Display 3, it will be found that an object (segment HI) closer to the mirror than your face needs a mirror of which size is larger than the half of the length (segment HI) because the ratio GE/HI exceeds 50 %, and similarly, that an object (segment JK) further from the mirror than your face needs a mirror of which size can be smaller that the half of the length (segment JK) because the ratio GE/JK is below 50 %.



(Display 3)

As expected at the inception, it is not a mere surmise that the size required for a mirror depends on its distance. In a special case where one sees the image of his/her face in a mirror, the size of the mirror is the half of the vertical length of one's face regardless of the its distance. That is, the fact always holds because the distance between you and the mirror is equal to that between your image and the mirror, regardless of the distance.

One point we should pay attention is that, the proposition "To see the image of your entire face, the size of the mirror is required at least half of the vertical length of your face" holds only in a special condition. That is, the mirror is plane, the mirror and your face are parallel, the size of your face means the vertical length of your face, the position of the mirror is fixed, and so forth. The last condition is very important and attracts our interest. Let's consider, for example, the case where you see the image of your whole body. When there is a mirror of which size is half of your height, it should be installed so that the top of the mirror is flush with the middle point between the top of your head and your eyes, and the bottom of the mirror is flush with the middle point between your eyes and your toe. In a room of a hotel or inn, there may be a mirror to see the image of your whole body, and in many cases the mirror is installed so that the bottom end of it is near to the floor. They may think that the mirror should be installed in the abovementioned manner to see the image of one's whole body. In fact, even if the mirror is installed so that the bottom end of it is flush with your waist, you can see the image of your whole body. This is one the advantages of mathematical explanation, in addition to the result of the experiment. In this manner mentioned above, we want to emphasize the merit of the mathematics in the class.

4. Modification of the condition: when the mirror is tilted

We have considered the Mirror problem under a special condition so far. Since we have been able to reach the conclusion in a simplified case, we will consider in a case under general conditions. Though the case where the traverse length of a mirror is focused may be interesting, we will consider a case where a mirror is tilted, here. That is, the problems are that the required size of a mirror is also half of the length of one's face when the mirror is tilted, and that the size of a mirror is also constant, and regardless of its distance when the mirror is tilted.

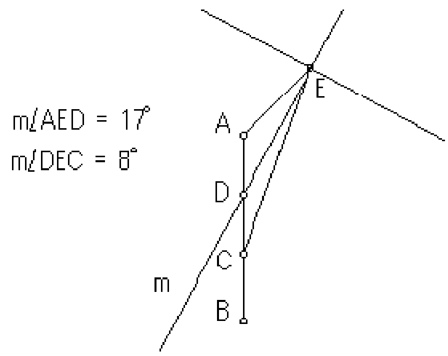
Let's construct a figure using the Sketchpad as before.

Step 1

Construct a segment AB, which corresponds to the vertical length of one's face, and a point C on segment AB, which corresponds to one's eye.

Step 2

Construct a line l, which corresponds to a mirror.

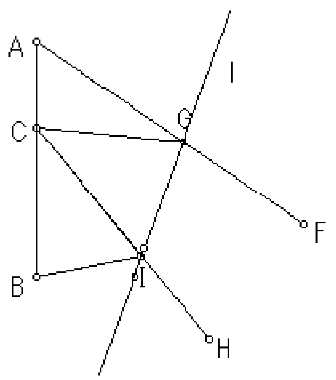


(Display 4)

Here comes up a problem that how to construct a line l so that the incident angle is equal to the angle of reflection. At first, let's construct as follows.

"Construct a middle point D on the segment AC, a perpendicular m to line l, and let the crossing be E. Construct lines connecting A and E, and C and E.

If we take measures of ∠AED and ∠DEC and move line l, we will find that the two angles are not equal as shown in Display 4. With this procedure of construction, the incident angle is not equal to the angle of reflection. In the case where the mirror and the face are parallel, we can construct a figure successively, but when the mirror is tilted, we cannot. Therefore, construct a figure as follows. (Refer to Display 5.)



(Display 5)

Step 3

Specify the line l as a symmetry axis, and construct a point F, which is the image of point C with respect to the symmetry axis. Construct a line connecting points A and F, and let the crossing with line l be G. Construct lines connecting A and G, and C and G.

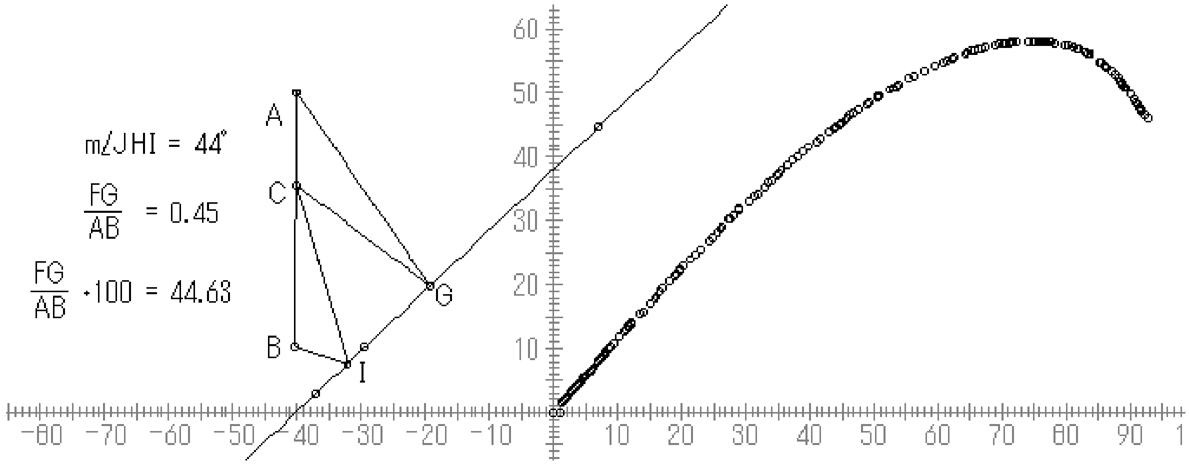
Step 4

Specify the line l as a symmetry axis, and construct a point H, which is the image of point B with respect to the symmetry axis. Construct a line connecting points C and H, and let the crossing with line l be I. Construct lines connecting C and I, and B and I.

Let's hide segment AF, point F, segment CH, and point H before consideration. It is possible to solve the problem by constructing and measuring the angle through which the mirror is tilted, and the distance between the mirror and the face, and then by tilting the mirror and changing the distance between the mirror and the face.

Regarding the problem that "Is the required size of a mirror also half of the length of one's face when the mirror is tilted?", we can reach the conclusion that the required size of the mirror can be smaller than the half of the length of one's face if the mirror is tilted. As the angle of the mirror to the horizontal line is reduced, the required size of the mirror approaches zero. It has to be noticed, however, that there is a limit to the angle of view for human eyes.

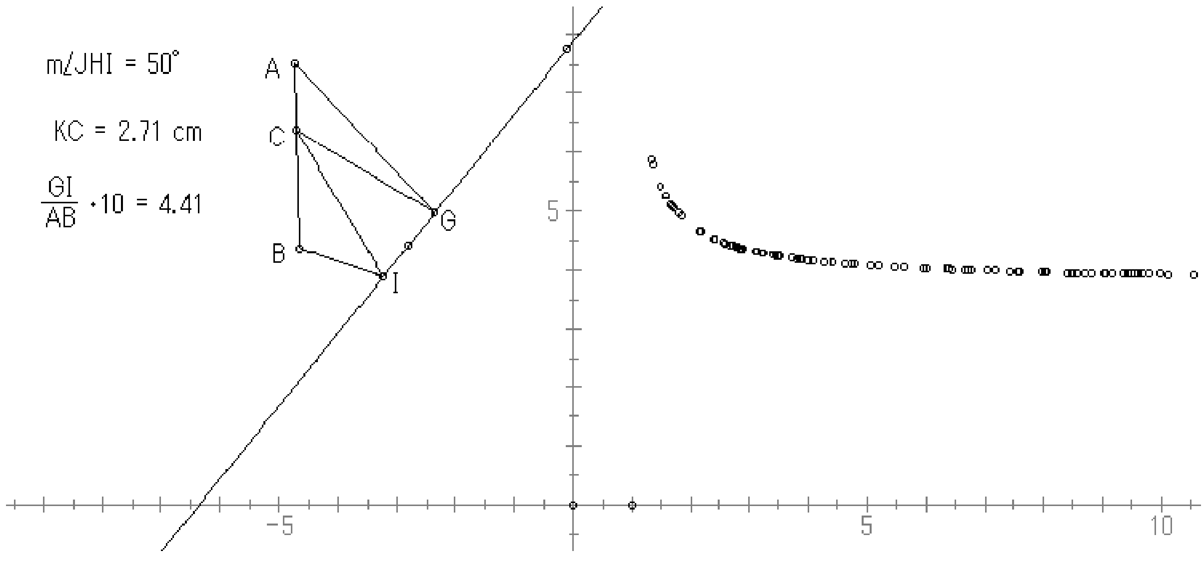
If we change the angles between the mirror and the horizontal line, how does the required size of the mirror change? We can plot the locus of a point J (X, Y) in a coordinate plane by specifying X for the angle between the mirror and the horizontal line, and Y for the required size of the mirror. As a result, the locus of point J is a curve similar to a sine curve as shown in Display 6.



(Display 6)

Regarding the problem that "Is the size of a mirror also constant, and regardless of its distance when the mirror is tilted?", we can change the distance between the mirror and the face by moving the mirror. When the mirror is translated parallelly, the required size of the mirror decreases. If x-axis represents the distance between your eye and the mirror, and y-axis, the ratio of the size of the mirror to the size of the face, the ratio converges to a certain value as shown in Display 7. The value is a little smaller than four (this value is ten times of the actual value because of the limited feature of the Sketchpad, therefore, 0.4 is the actual value). We can prove algebraically that the ratio of the size of the mirror to the size of the face approaches the limt 1/2 sin50 (=0.3830....).

This means that in the special case mentioned before where the angle is 90 degrees, that is, the mirror and the face are parallel, the ratio is 1/2, indicating the fact that the size of the mirror is half of the length of the face as proved before. Now, we can conclude that the required size of the mirror depends on its distance in general cases, and only when the face and the mirror are parallel, the size does not depend on its distance.



(Display 7)

Finally, I would like to thank Ms. Rika Kurai, a teacher of Yokohama Gakuin Girls' Senior/Junior High School, for her great assistance during the development of this educational material.