VISUAL UNDERSTANDING OF DEFINITIONS AND THEOREMS RELATING TO DIFFERENTIATION

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1. THE AIM OF GUIDANCE AND THE USE OF THE GRAPHIC CALCULATOR

When the students study functions, sequences, limits, differentiation, etc., it is difficult for them to get a visual understanding of the shape of a graph, how it changes, or how a limit is approached. Because definitions and theorems of functions, sequences, limits, differentiation, etc., are given to them without any preliminary knowledge, it is likely that the students can understand them only superficially. Here, we give them a visual image at the same time or prior to giving them its expression by using a graphic calculator. Then, we have the students try to derive theorems relating to functions, sequences, limits, differentiation, etc., and give them not only formulary but also visual understanding.

2. STARTING WITH DRAWING GRAPHS OF SEVERAL FUNCTIONS

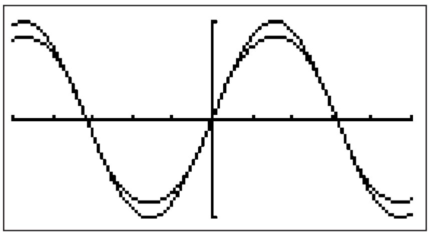
The students who have never used a graphic calculator before may be simply impressed by the fact that the graphic calculator can draw graphs of many functions. Therefore, you had better draw graphs of elementary functions such as linear functions, quadratic functions, y=sin x, y=cos x, $y=e^x$, y=log x, etc, at first then have them guess the shape of a function consisting of a combination of elementary functions. Let those students who can guess use the graphic calculator for confirmation, and those who cannot guess think about the shape after drawing the graph.
For example, have the students guess the shapes of the following functions. (Radian is used for angles)

$$y = \sin(\sin x) \qquad \cdots\cdots(1)$$

$$y = \frac{\sin x}{x} \qquad \cdots\cdots(2)$$

$$y = \frac{\sin\frac{1}{x}}{\frac{1}{x}} \qquad \cdots\cdots(3)$$

For comparison, we draw the graph of (1) and y=sin x in the same screen.

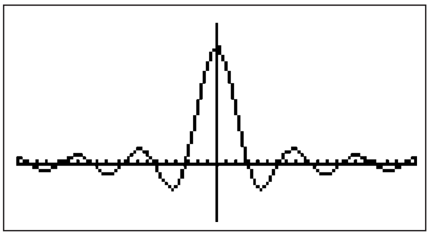


Screen 3

It can be found that the two graphs have the same period but different amplitudes. It is a good idea to have the students try to explain the reason why.

Next, have them guess the shape of the function (2).

Because -1<sin *x*<1, it can be guessed that y approaches zero when $x \to \infty$. However, it is not clear in the vicinity of the origin. As to this function, y is not defined at x=0. Then have the students think about the vicinity of the origin. We will obtain the following graph by using the graphic calculator.



Screen 4

Screen 4 indicates that y approaches 1 in the vicinity of zero and the following expression that is used to get derivatives of the elementary functions can be understood visually.

$$\lim_{x \to \infty} \frac{\sin x}{x} = 1 \qquad (4)$$

If some students ask why y approaches 1, have them try to prove it. Strictly speaking, however, this method using an area turns out to be a circular argument. Therefore, it can be one of the choices to have the students understand it intuitively.

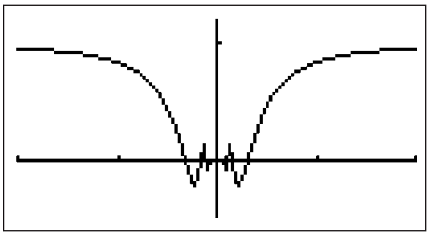
Rather than terminating our thinking here, we proceed to change the problem by substituting *x* with $\frac{1}{x}$. This is a graph of a function (3) and we obtain the following expression.

$$y = x \sin \frac{1}{x}$$

Because $\frac{1}{x}$ approaches zero by assigning larger and larger values to *x*, we find y approaches 1 by using a theorem (4).

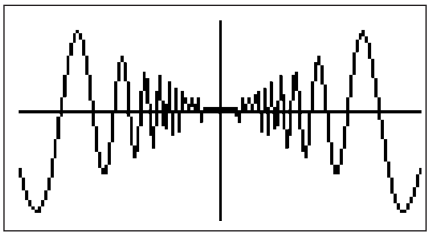
$$\lim_{x \to \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = 1$$

Similarly, it approaches zero near the origin. We can check this by drawing the graph using the graphic calculator.



Screen 5

This graph, however, shows a sort of chaos near the origin. Then, we enlarge the area near the origin using the Zoom Box feature. From the enlarged graph, we can find that the amplitude and period become smaller and smaller as x approaches the origin.



Screen 6

By defining y=0 at x=0, we can make this function continuous at the origin. However, the slope of a tangent fluctuates near the origin and we can conclude visually that the slope does not converge to a constant and the function is not differentiable at the origin. This means that the following expression proves to be an example of continuous but not differentiable functions.

$$y = x \sin \frac{1}{x}$$

It may be rather impossible to explain continuous but not differentiable functions, starting from having the students guess the shape of a function. But it is very difficult for them to understand the quality of continuity and differentiability without referring to any examples. It is important for them to draw many graphs in advance.

Reference

Yoshikazu Higuchi, Hiroshi Hosokawa, Toshikazu Ikeda : Fostering Student's Capacity in Mathematics, 1998, pp.123-135.

David Nelson, George Gheverghese Joseph, and Julian Williams : Multicultural Mathematics, Oxford University Press, 1993, pp.55-57.